Integrazione

$$\int 5\sqrt{x}\ dx$$

simbolo

differenziale

gli estremi di integrazione

$$\int_3^5 5\sqrt{x}\ dx$$

1 $[3, 5]$ ivi continua la funzione integranda quindi anche definita

$$\int_{-2}^{5} 5\sqrt{x}\ dx$$

$]-2, 0[$ non fa parte del dominio della funzione integranda

2 considerando un cambio di variabile

$\Rightarrow$ cambiamo

- il differenziale
- gli estremi di integrazione

$$\int \left[5\sqrt{x} + x^3 + \frac{1}{x} \right] dx =$$

per la linearità

$$= 5 \int \sqrt{x}\, dx + \int x^3\, dx + \int \frac{1}{x}\, dx =$$

$$\int x^{\alpha}\, dx = \frac{x^{\alpha+1}}{\alpha+1} + c \qquad \alpha \neq 0$$

$$D(x^n) = n \cdot x^{n-1} \quad \to \quad x^{n-1} = \frac{1}{n} D(x^n)$$

$$\int \frac{1}{x}\, dx \underset{x\neq 0}{=} \log |x| + c \underset{x\neq 0}{=} \ln |x| + c$$

$$D|\ln x| = \frac{1}{x} \qquad \left[D(\log_a x) = \frac{1}{x} \cdot \log_a e \right]$$

$$= 5\int x^{\frac{1}{2}}\,dx + \int x^{3}\,dx + \int \frac{1}{x}\,dx =$$

$$= 5\,\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{x^{3+1}}{3+1} + \ln|x| + c =$$ c = costante

$$= 5\,x^{3/2}\cdot\frac{2}{3} + \frac{x^{4}}{4} + \ln|x| + c$$

$$= \frac{10}{3}\,x^{3/2} + \frac{x^{4}}{4} + \ln|x| + c$$

- $$\int \frac{x}{\sqrt{a+x^{2}}}\,dx = \int\left(x\,\frac{2}{2}\,\underbrace{(a+x^{2})}\,^{-1/2}\right)dx = \frac{1}{2}\int 2x\,(a+x^{2})^{-1/2}\,dx =$$

$$D(a+x^{2}) = 2x$$

$$= \frac{1}{2}\int \underbrace{(a+x^{2})}_{t}{}^{-\frac{1}{2}}\,d\underbrace{(a+x^{2})}_{2x\,dx} = \frac{1}{2}\int t^{-1/2}\,dt = \frac{\cancel{1}}{\cancel{2}}\,\frac{(a+x^{2})^{-\frac{1}{2}+1}}{\cancel{1/2}} + C$$

$1/2$

- $\displaystyle\int \operatorname{ctg} x \, dx = \int \overbrace{\frac{\cos x}{\operatorname{sen} x} \, dx}^{d(\operatorname{sen} x)} = \int \frac{1}{\operatorname{sen} x} \cdot \overbrace{\cos x \, dx}^{D(\operatorname{sen} x)} =$

$\cos x = D(\operatorname{sen} x)$

$= \ln |\operatorname{sen} x| + c$

- $\displaystyle\int \operatorname{sen}(ax) \, dx = \int \overbrace{\frac{a}{a} \operatorname{sen} ax \, dx}^{D(ax)} = -\frac{1}{a} \cos(ax) + C$

$D(\cos x) = -\operatorname{sen} x$

$D(\cos(ax)) = -\operatorname{sen} ax \cdot a$

- $\displaystyle\int \operatorname{tg} x \, dx = \int \overbrace{\frac{\operatorname{sen} x}{\cos x} \, dx}^{-D(\cos x)} = -\ln |\cos x| + c$

$$\cdot \int \frac{(\arcsin x)^2}{\sqrt{1-x^2}}\,dx = \frac{(\arcsin x)^3}{3} + C$$

$$D(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$$

$$\cdot \int \frac{1}{\sqrt{a^2-x^2}}\,dx = \int \frac{1}{a\sqrt{1-\frac{x^2}{a^2}}}\,dx = \qquad \boxed{a>0}$$

$$D\left(\frac{x}{a}\right) = \frac{1}{a}$$

$$= \arcsin \frac{x}{a} + C$$

- $$\int \frac{1}{a^2+x^2}\,dx = \int \frac{1}{a^2\left(1+\frac{x^2}{a^2}\right)}\,dx = \qquad a>0$$

$$D(\operatorname{arctg} x) = \frac{1}{1+x^2}$$

$$D\left(\frac{x}{a}\right) = \frac{1}{a}$$

$$= \frac{1}{a}\int \frac{1}{1+\frac{x^2}{a^2}} \cdot \frac{1}{a}\,dx = \frac{1}{a}\operatorname{arctg}\frac{x}{a} + c$$

- $$\int \frac{1}{(1+x^2)\operatorname{arctg} x}\,dx = \int \frac{1}{\operatorname{arctg} x} \cdot \underbrace{\frac{1}{1+x^2}}_{D(\operatorname{arctg} x)}\,dx = \ln|\operatorname{arctg} x| + c$$

$$\frac{1}{t} = D(\ln|t|)$$

$$\cdot \int \frac{5}{x \log x} \, dx = 5 \int \frac{1}{\log(x)} \underbrace{\frac{1}{x} \, dx}_{D(\log x)} = \log |\log x| + c$$

$$\cdot \int \frac{(\log x)^n}{x} \, dx = \frac{(\log x)^{n+1}}{n+1} + c \qquad n \neq -1$$

$$\frac{1}{x} \, dx = D(\log x)$$

$$\cdot \int \frac{\overbrace{\sin 2x \qquad dx}^{D(\text{denominatore})}}{1 + (\sin x)^2} = \ln \left[\underbrace{\overbrace{1 + (\sin x)^2}^{[0,1]}}_{>0} \right] + c$$

non è necessario mettere il val. ass.

$$\sin 2x = 2 \sin x \cos x \qquad \int \frac{1}{t} \, dt = \ln|t|$$

$$D\left(1 + (\sin x)^2\right) = 2 \sin x \cdot \underbrace{\cos x}_{D(\sin x)}$$

$$\cdot \int 3x\, e^{x^2}\, dx \qquad \int 3 \cdot \frac{\overbrace{2x}^{D(x^2)}}{2}\, e^{x^2}\, dx = \frac{3}{2}\, e^{x^2} + c$$

$$D(e^x) = e^x$$

$$D\left(e^{x^2}\right) = e^{x^2} \cdot \underbrace{2x}_{D(x^2) = D(\text{esponente})}$$

$$\cdot \int \frac{1}{(5x+3)^6}\, dx = \int (5x+3)^{-6}\, \frac{\overbrace{5\, dx}^{D(5x+3)}}{5} = \frac{1}{5}\, \frac{(5x+3)^{-6+1}}{-6+1} + c =$$

$$D(5x+3) = 5$$

$$= \frac{1}{5} \cdot \frac{(5x+3)^{-5}}{-5} + c = -\frac{1}{25}\, \frac{1}{(5x+3)^5} + c$$

- $\displaystyle \int \sqrt{\frac{1+x}{1-x}}\,dx =$

$$\sqrt{\frac{1+x}{1-x}} \cdot \frac{\sqrt{1+x}}{\sqrt{1+x}} = \frac{\sqrt{(1+x)^2}}{\sqrt{(1-x)(1+x)}} = \frac{1+x}{\sqrt{1-x^2}}$$

$$= \int \left(\frac{1+x}{\sqrt{1-x^2}}\right) dx = \int \frac{1}{\sqrt{1-x^2}}\,dx + \int \frac{x}{\sqrt{1-x^2}}\,dx$$

additività degli integrali

$D(1-x^2)$

$$= \operatorname{arcsen} x + \int \frac{-2x}{-2} (1-x^2)^{-1/2}\,dx =$$

$-\sqrt{1-x^2}$

$$= \operatorname{arcsen} x - \frac{1}{2}\,\frac{(1-x^2)^{-1/2+1}}{-1/2+1} + C = \operatorname{arcsen} x - \cancel{\frac{1}{2}}\,\frac{(1-x^2)^{1/2}}{\cancel{1/2}} + C$$

$$\cdot \int \frac{e^{2x} - e^{x}}{e^{x} + 1}\, dx =$$

$$= \int e^{x}\, \frac{(e^{x} - 1)}{e^{x} + 1}\, dx = \int \frac{t - 1}{t + 1}\, dt =$$

pongo $e^{x} = t$ $\qquad$ $de^{x} = dt$

$e^{x} dx = dt$

$$= \int \frac{t \overbrace{+1 - 2}^{-1}}{t + 1}\, dt =$$

$$= \int \frac{\cancel{t+1}^{\,1}}{\cancel{t+1}}\, dt - 2 \int \frac{1}{t+1}\, dt =$$

$$= t - 2 \ln |t + 1| + c = e^{x} - 2 \ln (e^{x} + 1) + c$$

$t = e^{x}$

$$\int_2^3 \frac{e^{2x} - e^x}{e^x+1}\,dx =$$

$$= \ldots\ldots =$$

$e^x = t$ — cambio dei valori degli estremi di integrazione

$x = \log_e t$

$t_1 = e^2$ con $x = 2$

$t_2 = e^3$ con $x = 3$

$$= \int_{e^2}^{e^3} \frac{t-1}{t+1}\,dt = \ldots = \left[t - 2\ln|t+1| + c \right]_{e^2}^{e^3} =$$

le stesse cose

su e^x

$$= \left[e^x - 2\ln(e^x+1) \right]_2^3 = e^3 - 2\ln(e^3+1) - e^2 - 2\ln(e^2+1)$$

$$\int \frac{x\,dx}{1-\sqrt{x+1}} =$$

razionalizzazione

$$= \int \frac{x\,dx}{1-\sqrt{x+1}} \cdot \frac{1+\sqrt{x+1}}{1+\sqrt{x+1}} = \int \frac{x\,(1+\sqrt{x+1})}{\cancel{1}-x-\cancel{1}}\,dx =$$

$$= -\int \left(1+\sqrt{x+1}\right) dx = -\int \left(1+(x+1)^{1/2}\right) dx =$$

$$= -x - \frac{(x+1)^{1/2+1}}{1/2+1} + C =$$

$$= -x - \frac{2}{3}(x+1)^{3/2} + C$$

• $\displaystyle\int \frac{x-8}{\sqrt[3]{x}-2}\,dx = \int \frac{(x-8)}{\sqrt[3]{x}-2}\cdot\frac{\sqrt[3]{x^2}+4+2\sqrt[3]{x}}{\sqrt[3]{x^2}+4+2\sqrt[3]{x}}\,dx =$

RAZIONALIZZAZIONE → differenza di cubi (a^3-b^3)

$$= \int \frac{\cancel{x-8}}{\cancel{x-8}}\left[x^{2/3}+4+2x^{1/3}\right]dx = \frac{3}{5}x^{5/3}+4x+\frac{3}{\cancel{4}_2}\cdot\cancel{2}x^{4/3}+c$$

$\left(\frac{2}{3}+1\right)$ $(+1)$ $\left(\frac{1}{3}+1\right)$

$x^{5/3} = x\cdot\sqrt[3]{x^2}$ $\qquad x^{4/3} = x\sqrt[3]{x}$

• $\displaystyle\int \frac{1+\sin 2x}{\cos^2 x}\,dx =$

↑ additività

$\sin 2x = 2\sin x\cos x$

$$= \int \frac{1}{\cos^2 x}\,dx + \int \frac{2\sin x\,\overbrace{\cancel{\cos x}}^{-D(\cos x)}}{\cos^{\cancel{2}} x}\,dx = \operatorname{tg} x - 2\ln|\cos x| + c$$

$$D(\operatorname{tg} x) = \frac{1}{\cos^2 x} = 1+\operatorname{tg}^2 x$$

• $\int \cos x \, (\operatorname{tg} x + e^{\operatorname{sen} x}) \, dx =$

$\operatorname{tg} x \to \dfrac{\operatorname{sen} x}{\cos x}$

additività

D(sen x), esponente esponenziale

$$= \int \cancel{\cos x} \cdot \frac{\operatorname{sen} x}{\cancel{\cos x}} \, dx + \int \cos x \cdot e^{\operatorname{sen} x} \, dx = -\cos x + e^{\operatorname{sen} x} + c$$

• Raccolgo, scompongo

$$\int \frac{\operatorname{sen} x - \operatorname{sen}^3 x}{1 + \operatorname{sen} x} \, dx = \int \frac{\operatorname{sen} x \, (1 - \operatorname{sen}^2 x)}{1 + \operatorname{sen} x} \, dx = \int \frac{\operatorname{sen} x \, (1 - \operatorname{sen} x)\cancel{(1 + \operatorname{sen} x)}}{\cancel{1 + \operatorname{sen} x}} \, dx$$

moltiplico

$$= \int \operatorname{sen} x \, dx - \int \operatorname{sen}^2 x \, dx = (*)$$

$I = \int \operatorname{sen}^2 x \, dx =$

$\cos 2x = 1 - 2 \operatorname{sen}^2 x$

$$I = \int \operatorname{sen}^2 x \, dx = \frac{1}{2}\int (1-\cos 2x)\, dx = \frac{1}{2}\left(x - \frac{1}{2}\operatorname{sen} 2x\right) + c_1$$

$$\cos 2x = \begin{cases} \cos^2 x - \operatorname{sen}^2 x \\ 2\cos^2 x - 1 & \rightarrow \cos^2 x = \dfrac{\cos 2x + 1}{2} \\ 1 - 2\operatorname{sen}^2 x & \rightarrow \operatorname{sen}^2 x = \dfrac{1-\cos 2x}{2} \end{cases}$$

↑ 2° grado ↑ 1° grado

denominatore comune

$$= \frac{1}{4}\left(2x - \operatorname{sen} 2x\right) + c_1$$

$$(*) = -\cos x - I = -\cos x - \frac{1}{4}\left(2x - \operatorname{sen} 2x\right) + C$$

- $$\int \frac{x^3 + a^3}{x+a}\, dx \underset{\text{scompongo}}{=} \int \frac{\cancel{(x+a)}\,(x^2 + a^2 - ax)}{\cancel{x+a}}\, dx \underset{x \neq -a}{=}$$

$$= \int (x^2 + a^2 - ax)\, dx = \frac{x^3}{3} + a^2 x - \frac{a x^2}{2} + c$$

$$\bullet \int \frac{dx}{\sqrt{x+a}+\sqrt{x+b}}\,dx = \int \left\{ \frac{1}{\sqrt{x+a}+\sqrt{x+b}} \cdot \frac{\sqrt{x+a}-\sqrt{x+b}}{\sqrt{x+a}-\sqrt{x+b}} \right\} dx =$$

razionalizzare

differenza di quadrati

$$= \int \frac{\sqrt{x+a}-\sqrt{x+b}}{\cancel{x}+a-\cancel{x}-b}\,dx = \frac{1}{a-b}\left\{ \int \sqrt{x+a}\,dx - \int \sqrt{x+b}\,dx = \right.$$

$$= \frac{1}{a-b} \cdot \frac{2}{3}\left(\sqrt{(x+a)^3} - \sqrt{(x+b)^3} \right) + C$$

$$\int \sqrt{x}\,dx = \int x^{1/2}\,dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C_1 = \frac{x^{3/2}}{3/2} + C_1 = \frac{2}{3}\sqrt{x^3} + C_1$$