PROMEMORIA

COMPORTAMENTO ASINTOTICO ($\simeq$)

$e_{n\to+\infty}(n^3+3) \simeq e_n\, n^3$

[SI COMPORTA COME ... $n\to+\infty$]

es.1 $a_n = \log\left(1+\frac{1}{n}\right) \simeq \frac{1}{n}$

la prima successione si comporta come la seconda successione per $n\to+\infty$

COMPORTAMENTO ASINTOTICO

perché

$$\lim_{n\to+\infty} \frac{\log\left(1+\frac{1}{n}\right)}{\frac{1}{n}} = 1$$

es 2 $\quad \operatorname{sen}\frac{1}{n} \simeq \frac{1}{n}$

perché

$$\lim_{n\to+\infty} \frac{\operatorname{sen}\frac{1}{n}}{\frac{1}{n}} = 1$$

COMPORTAMENTO "INFINITESIMO"

$$\lim_{n \to +\infty} \frac{\log n}{n} = 0$$

$\log n$ è infinitesimo rispetto a n

posso scrivere

$$\log n = o(n) \leftarrow$$

Ricordatevi la "catena degli infiniti"

$$\log_a n < n^\alpha < a^n < n! < n^n$$

con $a > 1$ e $a \neq 1$

$\alpha > 0$ $\alpha \in \mathbb{R}$

$\Rightarrow$ il confronto tra questi comporta la "determinazione" degli "infinitesimi"

$$\sum_{n=1}^{+\infty} \frac{2n+1}{n^2(n+1)^2}$$ serie a termini positivi

termini generali : (della frazione)

$n \to +\infty$

$$\frac{2n+1}{n^2(n+1)^2} \sim \frac{2n}{n^2 \cdot n^2} = \frac{2n}{n^4} = 2\,\frac{1}{n^3}$$

$$\lim_{n\to+\infty} \frac{\frac{2n+1}{n^2(n+1)^2}}{\frac{2}{n^3}} = \lim_{n\to+\infty} \frac{\frac{2n\left(1+\overbrace{\frac{1}{2n}}^{\to 0}\right)}{n^4\left(1+\underbrace{\frac{1}{n}}_{\to 0}\right)^2}}{2/n^3} = 1$$

$$\Rightarrow \quad \frac{2n+1}{n^2(n+1)^2} \sim \frac{2}{n^3}$$ confronto asintotico

$$\sum_{n=1}^{+\infty} \frac{2}{n^3} = 2 \underbrace{\sum_{n=1}^{+\infty} \frac{1}{n^3}}$$

serie armonica generalizzata $\alpha = 3 > 1$ convergente

$$\sum_{n=1}^{+\infty} \frac{2n+1}{n^2(n+1)^2} \sim \sum_{n=1}^{+\infty} \frac{2}{n^3} = 2 \sum_{n=1}^{+\infty} \frac{1}{n^3}$$

per il criterio del confronto asintotico

convergente

reciproco del quadrato del numero

$$\sum_{k=1}^{+\infty} \frac{2k+1}{k^2 (k+1)^2} = \sum_{k=1}^{+\infty} \left[\frac{1}{k^2} - \frac{1}{(k+1)^2} \right]$$

reciproco del quadrato del successivo del numero

$$\frac{2k+1}{k^2(k+1)^2} = \frac{A}{k^2} + \frac{B}{(k+1)^2} = \frac{A(k+1)^2 + Bk^2}{k^2(k+1)^2} =$$

$$= \frac{A(k^2+2k+1) + B\,k^2}{k^2\,(k+1)^2} = \frac{A\,k^2 + 2Ak + A + B\,k^2}{k^2\,(k+1)^2} =$$

$$= \frac{k^2\,(A+B) + 2k\,A + A}{k^2\,(k+1)^2} = \frac{2k+1}{k^2\,(k+1)^2}$$

$$\begin{cases} A+B = 0 \\ 2A = 2 \\ A = 1 \end{cases}$$

equivalenti (ridondanze)

$$\rightarrow \begin{cases} A = 1 \\ B = -A = -1 \end{cases}$$

$$= \sum_{k=1}^{+\infty} \left[\frac{1}{k^2} - \frac{1}{(k+1)^2} \right] =$$

serie telescopica

$$= \sum_{k=1}^{+\infty} \left[b_1 - \frac{1}{(k+1)^2} \right] =$$

$\leftarrow$ somme

$$S_n = 1 - \frac{1}{(n+1)^2}$$

$$S = \lim_{n \to +\infty} \left[1 - \underbrace{\frac{1}{(n+1)^2}}_{\to 0} \right] = 1$$

SERIE TELESCOPICA

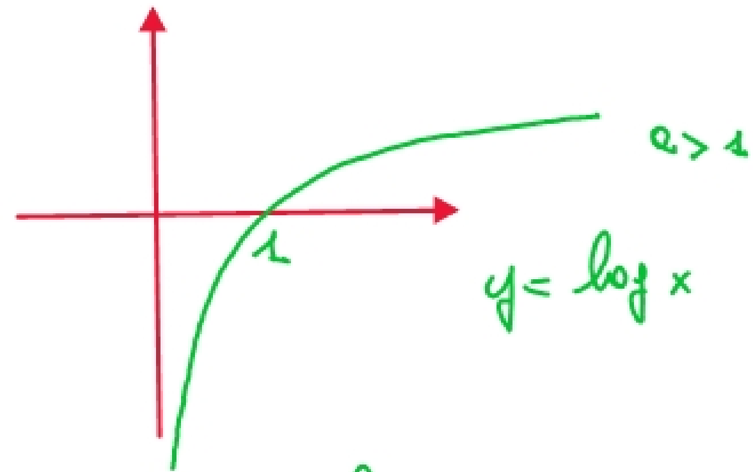
$$\sum_{n=0}^{+\infty} a_n = \sum_{k=0}^{+\infty} \left[b_k - b_{k+1} \right]$$

$$S = \lim_{n \to +\infty} \left(b_0 - b_{n+1} \right) = b_0 - \lim_{n \to +\infty} \left(b_{n+1} \right)$$

- $\sum_{n=2}^{+\infty} \log\left(1-\frac{1}{n^2}\right)$

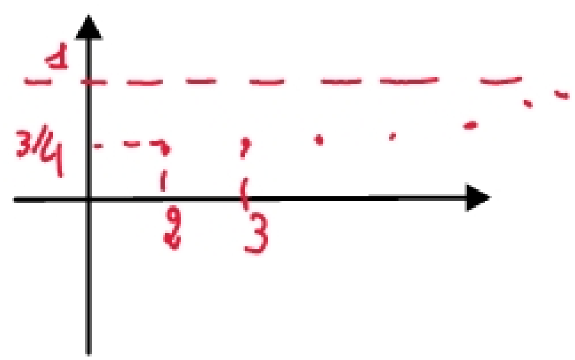
$$b_n = \log\left(1-\frac{1}{n^2}\right)$$

$$c_n = 1-\frac{1}{n^2} \qquad \lim_{n\to+\infty}\left(1-\frac{1}{n^2}\right) = 1$$



$$0 < 1-\frac{1}{n^2} < 1$$

$$0 < \text{Arg} < 1$$
$$\log(\text{Arg}) < 0$$



Successione

$$0 < 1-\frac{1}{n^2} < 1$$

SERIE A TERMINI NEGATIVI $\Leftarrow$ $\log\left(1-\frac{1}{n^2}\right) < 0$

Si considera la serie $\sum_{n=2}^{+\infty} \left[-\log\left(1-\frac{1}{n^2}\right) \right]$

considerando la successione $-\log\left(1-\frac{1}{n^2}\right) \sim \frac{1}{n^2}$

(serie a termini positivi

per il criterio del confronto asintotico

$$-\log\left(1-\frac{1}{n^2}\right) \sim \frac{1}{n^2}$$

di fatti $\lim_n \frac{-\log\left(1-\frac{1}{n^2}\right)}{\frac{1}{n^2}} = \lim_n \frac{\log\left(1-\frac{1}{n^2}\right)}{-\frac{1}{n^2}} = 1$

$\sum \frac{1}{n^2}$ serie armonica generalizzata $\alpha = 2 > 1$

$\Rightarrow \sum_{k=2}^{+\infty} \left\{ -\log\left(1-\frac{1}{n^2}\right) \right\}$ convergente

$$\Rightarrow \sum_{n=2}^{+\infty} \log\left(1-\frac{1}{n^2}\right)$$ converge perché converge la "sua opposta"

$$\sum_{n=2}^{+\infty} \left[-\log\left(1-\frac{1}{n^2}\right)\right]$$

Calcolo della somma della serie:

$$\underset{n=2}{\log\left(\frac{3}{4}\right)} + \underset{n=3}{\log\left(\frac{8}{9}\right)} + \underset{n=4}{\log\left(\frac{15}{16}\right)} + \underset{n=5}{\log\left(\frac{24}{25}\right)} + \cdots$$

$$\log 3 - \underbrace{\log 4}_{-2\log 2} + \log 8 - \log 9$$

$$-2\log 2 \quad + 3\log 2 \quad - - - -$$

in generale :

$$\log\left(1-\frac{1}{n^2}\right) = \log\left(\frac{n^2-1}{n^2}\right) = \log\left(\frac{(n-1)(n+1)}{n^2}\right) =$$

$$= \log\left(\frac{n+1}{n}\cdot\frac{n-1}{n}\right) = \log\frac{n+1}{n} + \log\frac{n-1}{n} = \log\frac{n+1}{n} + \log\left(\frac{n}{n-1}\right)^{-1}$$

$$= \log\frac{n+1}{n} - \log\frac{n}{n-1}$$

serie telescopica

attenzione!

$$\sum_{n=2}^{+\infty} \log\left(1-\frac{1}{n^2}\right) = -\log 3/4 + \log\frac{n+1}{n}$$

(attenzione se si calcola a partire dalla serie a termini positivi o a termini negativi)

$$S = \lim_n S_n = \lim_n\left(-\log 3/4 + \underbrace{\log\frac{n+1}{n}}_{\to \log 1 = 0}\right) =$$

$$= -\log 3/4$$

CRITERIO DELLA RADICE

• $\sum_{n=1}^{+\infty} \left(\frac{2n+1}{2n-1}\right)^{n}$ serie a termini positivi

$$\lim_{n\to+\infty} \sqrt[n]{\left(\frac{2n+1}{2n-1}\right)^{n}} = \lim_{n\to+\infty} \frac{2n+1}{2n-1} = \frac{2}{2} = 1$$ INEFFICACE?

• $\sum_{n=1}^{+\infty} \left(\frac{n+1}{2n-1}\right)^{n}$ termini positivi

$$\lim_{n\to+\infty} \sqrt[n]{\left(\frac{n+1}{2n-1}\right)^{n}} = \lim_{n} \frac{n+1}{2n-1} = \frac{1}{2} < 1$$ OK converge

$$\cdot \sum_{n=1}^{+\infty} \left(\frac{n}{3n-1}\right)^{2n-1}$$

positivi num, den, esp $\Rightarrow$
serie a termini positivi

$$\lim_{n\to+\infty} \sqrt[n]{\left(\frac{n}{3n-1}\right)^{2n-1}} =$$

$$= \lim_{n} \left(\frac{n}{3n-1}\right)^{\frac{2n-1}{n}} = \lim_{n} \left(\frac{n}{3n-1}\right)^{2-\frac{1}{n}} =$$

$$= \lim_{n} \left\{ \left(\frac{n}{3n-1}\right)^{2} \left(\frac{n}{3n-1}\right)^{-\frac{1}{n}} \right\} =$$

$$= \left(\frac{1}{3}\right)^{2} \cdot \left(\frac{1}{3}\right)^{0} = \frac{1}{9} < 1 \quad \text{converge}$$

- $$\sum_{n=1}^{+\infty} \left(\frac{n}{n+1}\right)^{n^2}$$ serie a termini positivi

$$\lim_{n\to+\infty} \sqrt[n]{\left(\frac{n}{n+1}\right)^{n^2}} = \lim_{n} \left(\frac{n}{n+1}\right)^{\frac{n^2}{n}} =$$

$$= \lim_{n} \left(\frac{n}{n+1}\right)^{n} = \lim_{n} \left(\frac{n+1}{n}\right)^{-n} =$$

$$= \lim_{n} \left[\left(1+\frac{1}{n}\right)^{n}\right]^{-1} = e^{-1} = \frac{1}{e} < 1 \quad \text{conv.}$$

- $$\sum_{n=1}^{+\infty} \frac{1}{\ln^n (n+1)}$$ Termini positivi

$$\lim_{n\to+\infty} \sqrt[n]{\frac{1}{\ln^n (n+1)}} = \lim_{n} \frac{1}{\underbrace{\ln(n+1)}_{\to+\infty}} = 0 < 1 \quad \text{conv.}$$

• $\displaystyle\sum_{n=1}^{+\infty} \frac{\left(\frac{n+1}{n}\right)^{n^2}}{3^n}$ termini positivi

$$\lim_n \sqrt[n]{\frac{\left(\frac{n+1}{n}\right)^{n^2}}{3^n}} = \lim_n \frac{\left(\frac{n+1}{n}\right)^{\frac{n^2}{n}}}{3^{\frac{n}{n}}} =$$

$$= \frac{1}{3} \lim_n \left(\frac{n+1}{n}\right)^n = \frac{1}{3} \lim_n \left(1+\frac{1}{n}\right)^n = \frac{e}{3} < 1$$

conv.

$2 < e < 3$

• $\displaystyle\sum_{1}^{+\infty} \frac{3^{n^2}}{(n!)^n}$ termini positivi

$$\lim_n \sqrt[n]{\frac{3^{n^2}}{(n!)^n}} = \lim_n \frac{3^{n^2/n}}{(n!)^{n/n}} = \lim_n \frac{3^n}{n!} \to 0 < 1$$

conv.

verificare le condizione necessarie

e_n è infinitesima ?

$$\lim_n \frac{3^{n^2}}{(n!)^n} = \lim_n \left(\frac{3^n}{n!}\right)^n =$$

$$= \lim_n e^{\log\left(\frac{3^n}{n!}\right)^n} =$$

$$= \lim_n e^{\overset{\to+\infty}{n} \, \overset{\to \log 0 = -\infty}{\log\left(\frac{3^n}{n!}\right)}} = e^{-(\infty)^2} = e^{-\infty} = 0$$