INTEGRAZIONE PER PARTI

DERIVAZIONE DEL PRODOTTO: $\int D(f(x)\cdot g(x))\,dx = \int f'(x)\cdot g(x)\,dx + \int f(x)\cdot g'(x)\,dx$

$$\int f(x)\cdot g'(x)\,dx = f(x)\cdot g(x) - \int f'(x)\cdot g(x)\,dx$$

funzione derivata

funzione da derivare

- $$\int \underbrace{(x^2-2x+3)}_{g'(x)}\,\underbrace{\ln(x)}_{f(x)}\,dx =$$

derivare il log. $\quad D(\ln x) = 1/x$

derivata di un polinomio

calcolo la primitiva

$$= \underbrace{\left(\frac{x^3}{3} - x^2 + 3x\right)}_{g(x)}\cdot \ln(x) - \int \frac{1}{x}\left(\frac{x^3}{3} - x^2 + 3x\right)dx =$$

$$= \left(\frac{x^3}{3} - x^2 + 3x\right) \ln(x) - \int \left(\frac{x^2}{3} - x + 3\right) dx =$$

$$= \left(\frac{x^3}{3} - x^2 + 3x\right) \ln(x) - \left[\frac{x^3}{9} - \frac{x^2}{2} + 3x\right] + C =$$

$$= \left(\frac{x^3}{3} - x^2 + 3x\right) \ln(x) - \frac{x^3}{9} + \frac{x^2}{2} - 3x + C$$

$$\bullet \int \underbrace{x}_{\text{de integrare}} \cdot \underbrace{\ln\left(\frac{1-x}{1+x}\right)}_{\text{de derivare}} dx = \frac{x^2}{2} \ln\left(\frac{1-x}{1+x}\right) - \int \frac{x^2}{2} \cdot \frac{1}{\frac{1-x}{\cancel{1+x}}} \cdot \frac{-1\cancel{-x}-1\cancel{+x}}{(1+x)^{\cancel{2}}} dx =$$

$$= \frac{x^2}{2} \ln \frac{1-x}{1+x} - \int \frac{x^2}{\cancel{2}} \frac{-\cancel{2}}{(1-x)(1+x)} dx =$$

$$= \frac{x^2}{2} \ln \frac{1-x}{1+x} + \int \frac{x^2}{(1-x)(1+x)} dx =$$

$$\int \frac{x^2}{(1-x)(1+x)}\,dx = \int \frac{x^2}{1-x^2}\,dx = -\int \frac{x^2}{x^2-1}\,dx$$

NUM e DEN sono di pari grado

per semplificare, riduco, la frazione:

$$= -\int \frac{x^2-1+1}{x^2-1}\,dx = -\int \left(\overset{1}{\cancel{\frac{x^2-1}{x^2-1}}} + \frac{1}{x^2-1} \right) dx =$$

$$= -x - \int \frac{1}{x^2-1}\,dx$$

↑ ✓ irriducibile, il grado del N. è < grado del D.

✓ il denominatore è scomponibile

scompongo il denominatore

$$(*)\quad \frac{1}{x^2-1} = \frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1} =$$

$$= \frac{A(x+1) + B(x-1)}{(x-1)(x+1)} = \frac{Ax + A + Bx - B}{(x+1)(x-1)} =$$

$$= \frac{x(A+B) + (A-B)}{(x+1)(x-1)}$$

per identità "di frazione" o "polinomiale"

$$\begin{cases} A + B = 0 \\ A - B = 1 \end{cases} \qquad \begin{cases} A = \frac{1}{2} \\ B = -A = -1/2 \end{cases}$$

$$(*) = \frac{1/2}{x-1} + \frac{-1/2}{x+1}$$

quindi

$$\int \frac{1}{x^2-1}\,dx = \int \left[\frac{1/2}{x-1} - \frac{1/2}{x+1} \right] dx =$$

$$= \frac{1}{2} \left(\ln|x-1| - \ln|x+1| \right) + c_1$$

Riprendendo l'integrale completo:

$$= \frac{x^2}{2} \ln\left(\frac{1-x}{1+x}\right) - x - \frac{1}{2} \ln\left|\frac{x-1}{x+1}\right| + c$$

$$\cdot \int \frac{\ln^2 x}{x^2}\,dx = \int \frac{1}{x^2}\cdot \ln^2 x\,dx =$$

$$\downarrow \quad \frac{1}{x^2} = D\left(-\frac{1}{x}\right)$$

$$= -\frac{\ln^2 x}{x} - \int -\frac{1}{x}\cdot \overbrace{2\cdot\frac{\ln x}{x}}^{D(\ln^2 x)}\,dx =$$

$$= -\frac{\ln^2 x}{x} + 2\int \frac{\ln x}{x^2}\,dx = -\frac{\ln^2 x}{x} + 2\int \frac{1}{x^2}\cdot \ln x\,dx =$$

$$= -\frac{\ln^2 x}{x} + 2\left\{ -\frac{1}{x}\cdot \ln x - \int -\frac{1}{x}\cdot \overbrace{\frac{1}{x}}^{D(\ln x)}\,dx \right. =$$

$$= -\frac{\ln^2 x}{x} - \frac{2}{x}\ln x + 2\int \frac{1}{x^2}\,dx =$$

$$= -\frac{1}{x}\ln^2 x - \frac{2}{x}\ln x - \frac{2}{x} + C$$

$$\bullet \int \frac{\ln(\ln x)}{x}\,dx = \int \ln(\ln x)\cdot d(\ln x) =$$

sostituisco $\underbrace{t = \ln x}$

$$\frac{1}{x}\,dx = D(\ln x)$$

$$= \int 1\cdot \ln t \; dt \underset{\text{p.p.}}{=} t\cdot \ln t - \int \cancel{t}\cdot \underbrace{\frac{1}{\cancel{t}}}_{D(\ln t)} dt =$$

$\int 1\,dt = t$

$$= t\cdot \ln t - t + c =$$

sostituendo la variabile

$$= \ln x \cdot \ln(\ln x) - \ln x + c =$$

$$= \ln x\,\big(\ln(\ln x) - 1\big) + c$$

$$\cdot \int x^2 \operatorname{arctg}(3x)\,dx = \frac{x^3}{3}\operatorname{arctg}(3x) - \int \frac{x^3}{\cancel{3}} \cdot \underbrace{\frac{1}{1+9x^2}\cdot\cancel{3}}_{D\operatorname{arctg}(3x)}\,dx =$$

$$x^2 = D\left(\frac{x^3}{3}\right)$$

$$= \frac{x^3}{3}\operatorname{arctg}(3x) - \int \frac{x^3}{1+9x^2}\,dx$$

$$\frac{x^3}{1+9x^2} = \frac{9x^3 + x - x}{9(1+9x^2)} = \frac{x(9x^2+1) - x}{9(1+9x^2)} =$$

$$= \frac{x}{9} - \frac{x}{9(1+9x^2)} \to$$

$$\int \frac{x^3}{1+9x^2}\,dx = \int \left\{\frac{x}{9} - \frac{x}{9(1+9x^2)}\right\} dx = \frac{x^2}{18} - \frac{1}{9}\cdot\frac{1}{18}\int\frac{18x}{1+9x^2}\,dx =$$

$$D(1+9x^2) = 18x$$

$$= \frac{x^2}{18} - \frac{1}{9} \cdot \frac{1}{18} \ln\left(1+9x^2\right) + C_1 =$$

$$= \frac{1}{18}\left(x^2 - \frac{1}{9} \ln\left(1+9x^2\right)\right) + C_2 \qquad \left(= \int \frac{x^3}{1+9x^2}\, dx \right)$$

Ricostruendo l'integrale originale:

$$\int x^2 \operatorname{arctg}(3x)\, dx = \frac{x^3}{3} \operatorname{arctg}(3x) - \frac{1}{18}\left(x^2 - \frac{1}{9} \ln\left(1+9x^2\right)\right) + C$$

$\bullet \int (\arcsin x)^2 \, dx =$

sostituzione $\quad t = \arcsin x$

$\qquad\qquad\qquad\qquad \downarrow$

$\qquad\qquad\qquad\qquad x = \sin t$

$\qquad\qquad\qquad\qquad dx = \cos t \, dt$

$$= \int t^2 \underbrace{\cos t \, dt}_{\cos t = D(\sin t)} \underset{p.p.}{=} t^2 \cdot \sin t - \int 2t \cdot \underset{\downarrow \, -\cos t}{\sin t} \, dt \underset{p.p.}{=}$$

$$= t^2 \sin t - 2\left\{ t(-\cos t) - \int \overset{D(t)}{1} (-\cos t) \, dt \right\} =$$

$$= t^2 \sin t + 2t \cos t - 2 \sin t + c$$

- $\displaystyle\int \frac{\arcsin x}{x^2}\,dx \underset{p.p.}{=} -\frac{1}{x}\arcsin x - \int -\frac{1}{x}\cdot\frac{1}{\sqrt{1-x^2}}\,dx =$

$+\frac{1}{x^2} = D\left(-\frac{1}{x}\right)$

$$= -\frac{1}{x}\arcsin x + \underbrace{\int \frac{1}{x\sqrt{1-x^2}}\,dx}_{I(x)}$$

$$I(x) = \int \frac{1}{x\sqrt{1-x^2}}\,dx = \int \frac{1}{\sin t\cdot\cancel{\cos t}}\cdot\cancel{\cos t}\,dt =$$

$x = \sin t$ $\qquad dx = \cos t\,dt$

$\sqrt{1-x^2} = \sqrt{1-\sin^2 t} = \cos t$ $\qquad \left|\frac{1}{\sin t} - \frac{\cos t}{\sin t}\right| = \left|\frac{1-\cos t}{\sin t}\right|?$

$$= \int \frac{1}{\sin t}\,dt = \Rightarrow ?\ \ln\left|\frac{1}{\sin t} - \operatorname{ctg} t\right| + c_1 .$$

$$\cdot \int \frac{\sin^2 x}{e^x}\, dx = \int e^{-x} \sin^2 x \, dx$$

$$e^{-x} = D\left(-e^{-x}\right)$$

$$= -e^{-x}\sin^2 x - \int -e^{-x} \cdot \underbrace{2\sin x \cos x}\, dx =$$

$$= -e^{-x}\sin^2 x + \overbrace{\int e^{-x} \underbrace{\sin 2x}\, dx}^{J(x)} =$$

$$\sin 2x = D\left(-\frac{\cos 2x}{2}\right)$$

$$\Rightarrow -\frac{e^{-x}}{2}\cos 2x - \int -e^{-x} \cdot \left(-\frac{\cos 2x}{2}\right) dx =$$

$$-\frac{e^{-x}}{2}\cos 2x - \frac{1}{2}\int e^{-x} \underbrace{\cos 2x}\, dx$$

$$D\left(\frac{\sin 2x}{2}\right)$$

$$\Rightarrow \int e^{-x}\cos 2x \, dx \underset{pp.}{=} \frac{1}{2}\,\mathrm{sen}\,2x\; e^{-x} - \underbrace{\int \frac{1}{2}\,\mathrm{sen}\,2x \cdot \left(-e^{-x}\right) dx}_{J(x)}$$

Sostituendo nella precedente uguaglianza

$\Rightarrow$ l'integrale diventa =

$$\left| \; J(x) = -\frac{e^{-x}}{2}\cos 2x - \frac{e^{-x}}{4}\,\mathrm{sen}\,2x - \frac{1}{4}J(x) \; \right|$$

$$\underbrace{J(x) = -\frac{e^{-x}}{\cancel{25}\,5}\left(2\cos 2x + \mathrm{sen}\,2x\right)}_{} + c_1$$

$\Rightarrow$ l'integrale:

$$\int \frac{\sin^2 x}{e^x}\, dx = -e^{-x}\sin^2 x - \frac{e^{-x}}{5}\left(2\cos 2x + \mathrm{sen}\,2x\right) + C$$