

Lezione 13

Coeficiente del fattore di una serie:

$$\sum_{h=1}^{+\infty} \frac{2^{h+1}}{(2h)!} = \sum_{h=1}^{+\infty} \frac{2^h}{(2h)!} + \sum_{h=1}^{+\infty} \frac{1}{(2h)!}$$

lineare

$$(2) \sum_{h=1}^{+\infty} \frac{1}{(2h)!} = \sum_{h=1}^{+\infty} \frac{1}{(2h)!} = \boxed{\cosh 1 - 1}$$

$$\sum_{h=0}^{+\infty} \frac{x^{2h}}{(2h)!} = \cosh x \quad \rightarrow x=1$$

$$a_0 = \cosh 0 = 1$$

$$a_0 = \frac{1^0}{0!} = 1$$

$$\begin{aligned}
 (2) \quad \sum_{n=1}^{+\infty} \frac{2^n}{(2n)!} &= \sum_{n=1}^{+\infty} \frac{2^n}{2^n (2n-1)!} = \\
 &= \sum_{n=1}^{+\infty} \frac{1}{(2n-1)!} =
 \end{aligned}$$

$$\sum_{n=0}^{+\infty} \frac{x^{2n+1}}{(2n+1)!} = \sinh x$$

pongo : $2n-1 = 2m+1$

$$2n = 2m+2$$

$$\begin{aligned}
 &= \sum_{m=0}^{+\infty} \frac{1}{(2m+1)!} = \\
 &\quad x=1
 \end{aligned}$$

$$= \sum_{m=0}^{+\infty} \frac{1}{(2m+1)!} = \sinh 1$$

$$\left. \begin{aligned}
 n=1 \\
 n=m+1 \\
 1=m+1 \\
 0=m
 \end{aligned} \right\} \text{ cambio } \text{Indice}$$

$$\rightarrow \sum_{n=1}^{+\infty} \frac{2^{n+1}}{(2n)!} = \sum_{n=1}^{+\infty} \frac{2^n}{(2n)!} + \sum_{n=1}^{+\infty} \frac{1}{(2n)!} =$$

$$= \sinh 1 + \cosh 1 - 1 =$$

$$= \frac{e^1 - e^{-1}}{2} + \frac{e^1 + e^{-1}}{2} - 1 =$$

$$= \frac{e^1 - \cancel{e^{-1}} + e^1 + \cancel{e^{-1}} - 2}{2} = e - 1$$

$$\begin{aligned} e^1 &= e \\ e^{-1} &= 1/e \end{aligned}$$

$$\bullet \sum_{h=1}^{+\infty} (-1)^h \frac{h+1}{h^2+1}$$

? convergenza assoluta

? convergenza semplice

1) Conv. assoluta

$$\sum_{h=1}^{+\infty} \left| (-1)^h \frac{h+1}{h^2+1} \right| = \sum_{h=1}^{+\infty} \frac{h+1}{h^2+1}$$

$$\lim_{h \rightarrow +\infty} \frac{h+1}{h^2+1} = 0$$

successione è infinitesima
(CN verificato)

confronto asintotico

$$a_n = \frac{h+1}{h^2+1} = \frac{h \left(1 + \frac{1}{h}\right)}{h^2 \left(1 + \frac{1}{h^2}\right)} \sim \frac{1}{h}$$

$$\sum_{h=1}^{+\infty} \frac{h+1}{h^2+1}$$

confronto asintotico

$$\sum_{h=1}^{+\infty} \frac{1}{h}$$

che è divergente
(serie armonica)

- $\sum_{n=1}^{+\infty} \frac{|n|}{n^2+1}$ è divergente $\Rightarrow \sum_{n=1}^{+\infty} (-1)^n \frac{n+1}{n^2+1}$ non
 converge assolutamente

2) Convergenza $\sum_{n=1}^{+\infty} (-1)^n \frac{n+1}{n^2+1}$

critero L: $\sum_{n=1}^{+\infty} \left| (-1)^n \frac{n+1}{n^2+1} \right| \rightarrow b_n = \frac{n+1}{n^2+1}$
 ① infinitesime $\lim_{n \rightarrow +\infty} \frac{n+1}{n^2+1} = 0$

② decrescente $\frac{N}{n+1} < \frac{D}{n^2+1}$
 $n < n^2 \quad \forall n > 1$
 $\frac{n+1}{n^2+1} \searrow$ vero $n \in \mathbb{N}$

$$\bullet \sum_{n=1}^{+\infty} \frac{1}{\sqrt{n} + \log n^3}$$

serie a termini positivi

convergenza assoluta :

$$\sum_{n=1}^{+\infty} \left| \frac{1}{\sqrt{n} + \log n^3} \right|$$

$$\frac{1}{\sqrt{n} + \log n^3} = \frac{1}{\sqrt{n} + 3 \log n} = \frac{1}{\sqrt{n} \left(1 + \frac{3 \log n}{\sqrt{n}} \right)} \sim \frac{1}{\sqrt{n}} - \frac{1}{n^{3/2}}$$

$\sum \frac{1}{n^{1/2}}$ serie armonica generalizzata con $\alpha = \frac{1}{2} < 1$

diverge

non ho la convergenza assoluta —

$$\cdot \sum_{n=1}^{+\infty} \frac{\cos(n+1)\pi}{\sqrt{n} + \log n^3} = \sum_{n=1}^{+\infty} \frac{(-1)^n}{\sqrt{n} + \log n^3}$$

$$\cos(n+1)\pi = \begin{cases} +1 \\ -1 \end{cases}$$

serie a segni alterni

✓ la conv. assoluta

$$\sum \left| \frac{(-1)^n}{\sqrt{n} + \log n^3} \right| = \sum \frac{1}{\sqrt{n} + \log n^3}$$

(vedi sopra)

No conv. ass.

✓ C.R.

$$\lim_{n \rightarrow +\infty} \frac{1}{\sqrt{n} + 3 \log n} = 0$$

$$a_n \geq a_{n+1} \quad (\text{decreciente})$$

$$\frac{1}{\sqrt{n} + 3 \log n} > \frac{1}{\sqrt{n+1} + 3 \log(n+1)}$$

Esempi di serie che convergono assolutamente:

$$\sum_{n=0}^{+\infty} x^n = \frac{1}{1-x} \quad |x| < 1$$

esempio $\sum_{n=0}^{+\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$ (serie geometrica)

$$\sum_{n=0}^{+\infty} \frac{x^n}{n!} = e^x \quad \forall x \in \mathbb{R}$$

esempio $\sum_{n=0}^{+\infty} \frac{2^n}{n!} = e^2$

$$\sum_{n=0}^{+\infty} \frac{(-1)^n x^{2n}}{(2n)!} = \cos x \quad \forall x \in \mathbb{R}$$

esempio :

$$\sum_{n=0}^{+\infty} \frac{(-1)^n 4^{2n}}{(2n)!} = \cos 4$$

$$\sum_{n=0}^{+\infty} \frac{(-1)^n \left(\frac{\pi}{6}\right)^{2n}}{(2n)!} = \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

$$\sum_{n=0}^{+\infty} \frac{(-1)^{n+1} x^{2n+1}}{(2n+1)!} = \sin x \quad \forall x \in \mathbb{R}$$

esempio

$$\sum_{n=0}^{+\infty} \frac{(-1)^{n+1} \left(\frac{\pi}{3}\right)^{2n+1}}{(2n+1)!} = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$\sum_{n=0}^{+\infty} \frac{(-1)^{n+1} \underbrace{5}_{\text{circled}} \left(\frac{\pi}{4}\right)^{2n+1}}{(2n+1)!} =$$

$$= 5 \sum_{n=0}^{+\infty} \frac{(-1)^{n+1} \left(\frac{\pi}{4}\right)^{2n+1}}{(2n+1)!} = 5 \sin \frac{\pi}{4} = \frac{5\sqrt{2}}{2}$$

$$\sum_{h=1}^{+\infty} \frac{(-1)^{h+1} x^h}{h} = \log(1+x)$$

$$\forall x \in \mathbb{R} \\ -1 < x \leq 1$$

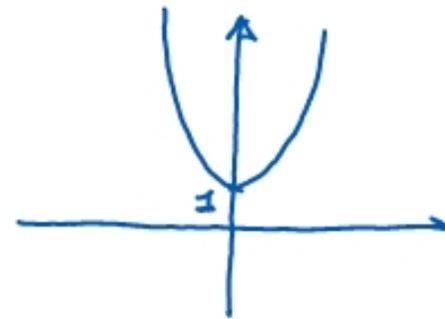
esempio

$$\sum_{h=1}^{+\infty} \frac{(-1)^{h+1} \frac{1}{4^h}}{h} = \log\left(1 + \frac{1}{4}\right) = \log \frac{5}{4}$$

$1 + \frac{1}{4} = \frac{5}{4}$

$$\sum_{h=0}^{+\infty} \frac{x^{2h}}{(2h)!} = \cosh x \quad \forall x \in \mathbb{R}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

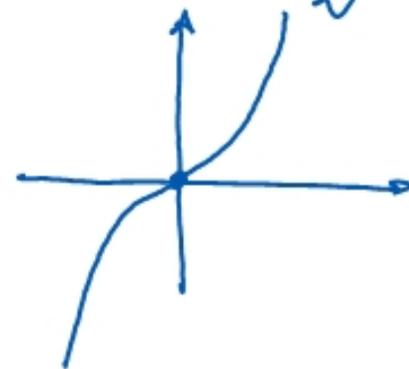


esempio

$$\sum_{h=0}^{+\infty} \frac{4^{2h}}{(2h)!} = \cosh 4 = \frac{e^4 + e^{-4}}{2}$$

$$\sum_{h=0}^{+\infty} \frac{x^{2h+1}}{(2h+1)!} = \sinh x \quad \forall x \in \mathbb{R}$$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$



esempio

$$\begin{aligned} \sum_{n=0}^{+\infty} \frac{\left(\frac{1}{2}\right)^{2n+1}}{(2n+1)!} &= \sinh \frac{1}{2} = \\ &= \frac{e^{1/2} - e^{-1/2}}{2} \end{aligned}$$

$$\cdot \sum_{h=1}^{+\infty} \frac{(-1)^h 2^{h+1}}{3^{h+1} h!} =$$

$$= \sum_{h=1}^{+\infty} (-1)^h \cdot \frac{2^{h+1}}{3^{h+1}} \cdot \frac{1}{h!} =$$

$$= \sum_{h=1}^{+\infty} (-1)^h \left(\frac{2}{3}\right)^{h+1} \cdot \frac{1}{h!} \cdot \frac{1}{3} = \sum_{h=0}^{+\infty} \frac{x^h}{h!} = e^x$$

$$= \frac{1}{3} \sum_{h=1}^{+\infty} \frac{(-1)^h \left(\frac{2}{3}\right)^{h+1}}{h!} = \frac{1}{3} \sum_{h=1}^{+\infty} \frac{(-1)^h \left(\frac{2}{3}\right)^h \cdot \frac{2}{3}}{h!} =$$

$$= \frac{1}{3} \sum_{h=1}^{+\infty} \frac{\left(-\frac{2}{3}\right)^h \cdot \frac{2}{3}}{h!} = \frac{2}{3} \sum_{h=1}^{+\infty} \frac{\left(-\frac{2}{3}\right)^h}{h!}$$

$$= \sum_{n=1}^{+\infty} \frac{\left(-\frac{2}{3}\right)^n}{n!} =$$

$$= \sum_{n=0}^{+\infty} \frac{\left(-\frac{2}{3}\right)^n}{n!} - \frac{\left(-\frac{2}{3}\right)^0 = 1}{\underbrace{0!}_{=1}} = \quad 0! = 1$$

$$= \sum_{n=0}^{+\infty} \left[e^{-2/3} - 1 \right]$$

$$x = -2/3$$

$$\sum_{n=2}^{+\infty} (-1)^n \frac{2^{n+1}}{3^{n+2} n!} = \dots =$$

$$= \frac{2}{9} \sum_{n=2}^{+\infty} \frac{(-2/3)^n}{n!} =$$

$$= \frac{2}{9} \left[\sum_{n=0}^{+\infty} \frac{(-2/3)^n}{n!} - \underbrace{\frac{(-2/3)^0}{0!}}_{e_0} - \underbrace{\frac{(-2/3)^1}{1!}}_{e_1} \right] =$$

$$= \frac{2}{9} \left(e^{-2/3} - 1 - (-2/3) \right) = \frac{2}{9} \left(e^{-2/3} - \frac{1}{3} \right)$$