Lezione 24: integrali

$$\int (x+1)^2 \cdot \cos x \, dx = (x+1)^2 \cdot \sin x - \int \underbrace{2(x+1)} \sin x \, dx =$$

da derivare $\quad$ integrare $\quad \cos x = D(\sin x)$

$$= (x+1)^2 \sin x - 2 \left\{ (x+1)(-\cos x) - \int (-\cos x) \, dx \right\} =$$

$$= (x+1)^2 \sin x + 2 \left\{ (x+1) \cos x - \int \cos x \, dx \right\} =$$

$$= (x+1)^2 \sin x + 2 \left\{ (x+1) \cos x - \sin x \right\} + C$$

$$\int 2x \,\mathrm{arctg}\, x \, dx = x^2 \,\mathrm{arctg}\, x - \int \frac{x^2}{x^2+1} \, dx =$$

$$D(\mathrm{arctg}\, x) = \frac{1}{x^2+1} \qquad = x^2 \,\mathrm{arctg}\, x - \int \frac{x^2+1-1}{x^2+1} \, dx =$$

$$\left(1 - \frac{1}{x^2+1}\right)$$

$$= x^2 \,\mathrm{arctg}\, x - x + \mathrm{arctg}\, x + C$$

$$\int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx =$$

$$= e^x \sin x - \left\{ e^x \cos x - \int e^x (-\sin x) \, dx \right\} =$$

$$= e^x \sin x - e^x \cos x - \int e^x \sin x \, dx$$

$$\int e^x \sin x \, dx = e^x (\sin x - \cos x) - \int e^x \sin x \, dx + C_1$$

$$\int e^x \sin x \, dx + \int e^x \sin x \, dx = e^x (\sin x - \cos x) + C_1$$

$$2 \int e^x \sin x \, dx = e^x (\sin x - \cos x) + C_1$$

$$\int e^x \sin x \, dx = \frac{e^x}{2} (\sin x - \cos x) + C$$

$$\int \sqrt{1-x^2}\,dx = x\sqrt{1-x^2} - \int x \cdot \frac{-\cancel{2}x}{\cancel{2}\sqrt{1-x^2}}\,dx =$$

$$= x\sqrt{1-x^2} - \int \frac{-x^2}{\sqrt{1-x^2}}\,dx =$$

$$= x\sqrt{1-x^2} - \int \frac{-x^2+1-1}{\sqrt{1-x^2}}\,dx =$$

$$= x\sqrt{1-x^2} - \int \left(\frac{1-x^2}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} \right) dx =$$

$$= x\sqrt{1-x^2} - \int \left(\sqrt{1-x^2} - \frac{1}{\sqrt{1-x^2}} \right) dx =$$

$$= x\sqrt{1-x^2} - \int \sqrt{1-x^2}\,dx + \arcsin x + C_1$$

$$\int \sqrt{1-x^2}\,dx = \frac{1}{2}\left(x\sqrt{1-x^2} + \arcsin x \right) + C$$

Funzioni Fratte

N.B. porre attenzione al grado del numeratore e al grado del denominatore

grado N $\geq$ grado D $\Longrightarrow$ FRAZIONARE!
RIDURRE
DIVIDERE N con D
$\Downarrow$
con: artifici di calcolo
divisione polinomiale
scomposizioni
- - - -

$$\frac{2x^2-3x+7}{x-5} = 2x+7+\frac{42}{x-5}$$

$$\begin{array}{rr|l}
\text{DIVIDENDO} & & \text{DIVISORE} \\
2x^2-3x+7 & & x-5 \\
\hline
-2x^2+10x & & 2x+7 \\
\cline{1-1}
// \quad +7x+7 & & \text{QUOZIENTE} \\
-7x+35 & & \\
\cline{1-1}
// \quad +42 & & \\
\text{RESTO} & &
\end{array}$$

$$(2x+7)(x-5)+42 = 2x^2-3x+7$$

$$\int \frac{2x^2-3x+7}{x-5}\,dx = \int \left(2x+7+\frac{42}{x-5}\right) dx =$$

$$= \int (2x+7)\,dx + 42\int \frac{1}{x-5}\,dx =$$

$$= x^2 + 7x + 42 \log|x-5| + c$$

1: grado N > grado D

2. il D è di 1° grado ($\Rightarrow$ non scomponibile)

$$\int \frac{3x-4}{x^2-6x+8}\,dx$$

1. grado N < grado D

2. denominatore è di 2° grado, è scomponibile
   (ovvero $x^2-6x+8=0 \Rightarrow x_1 \neq x_2 \in \mathbb{R}$)
   $\Delta > 0$

$\Rightarrow$ ottengo 2 termini di 1° grado

$$\frac{3x-4}{(x-4)(x-2)} = \frac{A}{x-4} + \frac{B}{x-2} = \frac{A(x-2)+B(x-4)}{(x-4)(x-2)} =$$

$$= \frac{(A+B)x-2A-4B}{(x-4)(x-2)}$$

IDENTITA' POLINOMIALE

$$3x-4 = (A+B)x - 2A - 4B$$

$$\begin{cases} A+B=3 \\ -2A-4B=-4 \end{cases} \qquad \begin{cases} A+B=3 \\ A+2B=2 \end{cases} \qquad \begin{cases} B=-1 \\ A=4 \end{cases}$$

$$A \qquad\qquad B$$

$$\frac{3x-4}{(x-4)(x-2)} = \frac{4}{x-4} + \frac{-1}{x-2}$$

$$= \int \frac{3x-4}{(x-4)(x-2)}\,dx = \int \left[\frac{4}{x-4} - \frac{1}{x-2} \right] dx =$$

$$= 4 \log|x-4| - \log|x-2| + c$$

$$\int \frac{3x}{x^3-1}\,dx$$

$(x-1)\,\underbrace{(x^2+x+1)}_{}$

falso quadrato

non scomponibile

$x^2+x+1=0 \qquad \Delta<0 \Rightarrow$ soluzioni in $\mathbb{C}$

→ METODO DI DECOMPOSIZIONE IN FRATTI SEMPLICI

↓

Den. è scomponibile

(*)

$$\frac{3x}{x^3-1} = \frac{3x}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1} =$$

$D(x^2+x+1) = 2x+1$ polinomio di 1° grado (*)

$$= \frac{A(x^2+x+1)+(Bx+C)(x-1)}{(x-1)(x^2+x+1)} = \frac{(A+B)x^2+x(A-B+C)+(A-C)}{(x-1)(x^2+x+1)}$$

Risoluzione sistema:

$$\begin{cases} A+B=0 \\ A-B+C=3 \\ A-C=0 \end{cases} \quad \begin{cases} B+C=0 \\ A-C=0 \\ A-B+C=3 \end{cases} \quad \begin{cases} B=-C \\ A=C \\ C+C+C=3 \end{cases} \quad \begin{cases} B=-1 \\ A=1 \\ C=1 \end{cases}$$

$$\int \frac{3x}{x^3-1}\,dx = \int \frac{1}{x-1}\,dx - \int \frac{x-1}{x^2+x+1}\,dx =$$

$$= \log|x-1| - \frac{1}{2}\int \frac{\overset{(2x+1)-3}{2x-2}}{x^2+x+1}\,dx =$$

$$= \log|x-1| - \frac{1}{2}\int \frac{2x+1}{x^2+x+1} + \frac{3}{2}\int \frac{1}{x^2+x+1}\,dx =$$

$$= \log|x-1| - \frac{1}{2}\log\left(x^2+x+1\right) + \underbrace{\frac{3}{2}\int \frac{1}{x^2+x+1}\,dx} =$$

$$D(\operatorname{arctg} t) = \frac{1}{1+t^2}$$

$$x^2 + x + 1 \longrightarrow k\left[1 + (\underbrace{ax+b}_{t})^2\right]$$

METODO DI "COMPLETAMENTO DEI QUADRATI"

$$x^2 + x + 1 = \left(x^2 + x + \frac{1}{4}\right) + \frac{3}{4} = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4} =$$

$$= \frac{3}{4}\left[1 + \frac{\left(x+\frac{1}{2}\right)^2}{3/4}\right] = \frac{3}{4}\left[1 + \left(\frac{x+\frac{1}{2}}{\sqrt{3}/2}\right)^2\right] =$$

$$= \frac{3}{4}\left[1 + \left(\frac{2x+1}{\sqrt{3}}\right)^2\right] = \frac{3}{4}\left[1 + \underbrace{\left(\frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}}\right)}_{\text{argomento dell' arctg } t}{}^2\right]$$

$$\frac{3}{2}\int \frac{1}{x^2+x+1}\,dx = \frac{3}{2}\int \frac{1}{\frac{3}{4}\left[1+\left(\frac{2}{\sqrt{3}}x+\frac{1}{\sqrt{3}}\right)^2\right]}\,dx =$$

$$= 2\int \frac{1}{\left[1+\left(\frac{2}{\sqrt{3}}x+\frac{1}{\sqrt{3}}\right)^2\right]}\,dx =$$

$$= 2\,\frac{\sqrt{3}}{2}\int \frac{\frac{2}{\sqrt{3}}}{\left[1+\left(\frac{2}{\sqrt{3}}x+\frac{1}{\sqrt{3}}\right)^2\right]}\,dx = \sqrt{3}\,\operatorname{arctg}\frac{2x+1}{\sqrt{3}} + c_1$$

$$\int \frac{3x}{x^3-1}\,dx = \log|x-1| - \frac{1}{2}\log\left(x^2+x+1\right) + \sqrt{3}\,\operatorname{arctg}\frac{2x+1}{\sqrt{3}} + c$$

$$\int \frac{9x+8}{x^3+2x^2+x+2}\,dx = \int \frac{9x+8}{(x+2)(x^2+1)}\,dx =$$

Ruffini, scomposizione $= x^2(x+2) + (x+2) = (x+2)(x^2+1)$

$$\frac{9x+8}{(x+2)(x^2+1)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+1} =$$

$$= \frac{A(x^2+1) + (Bx+C)(x+2)}{(x+2)(x^2+1)} =$$

$$= \frac{x^2(A+B) + x(2B+C) + A+2C}{(x+2)(x^2+1)} \Rightarrow$$

$$\begin{cases} A+B = 0 \\ 2B+C = 9 \\ A+2C = 8 \end{cases} \quad \dots \quad \begin{cases} A = -2 \\ B = 2 \\ C = 5 \end{cases}$$

$$= \int \frac{-2}{x+2}\,dx + \int \frac{2x}{x^2+1}\,dx + \int \frac{5}{x^2+1}\,dx =$$

$$= -2\log|x+2| + \log(x^2+1) + 5\,\mathrm{arctg}\,x + C =$$

$$= \log\frac{x^2+1}{(x+2)^2} + 5\,\mathrm{arctg}\,x + C$$

$$\int \frac{e^x}{e^{2x}+3e^x+2}\,dx =$$

posto $e^x = t$

$x = \log t$

SOSTITUZIONE

$dx = \frac{1}{t}\,dt$

$$= \int \frac{\cancel{t}}{t^2-3t+2}\cdot\frac{1}{\cancel{t}}\,dt = \int \frac{1}{t^2-3t+2}\,dt =$$

$$= \int \frac{1}{(t-1)(t-2)}\,dt$$

$$\frac{\quad}{|(\ -)|(\ 2|} = \frac{A}{(t-2)} + \frac{B}{(t-1)} = \frac{A(t-1)+B(t-2)}{(t-1)(t-2)} =$$

$$= \frac{(A+B)t - A - 2B}{(t-1)(t-2)}$$

$$\begin{cases} A+B=0 \\ -A-2B=1 \end{cases} \qquad \begin{cases} A=1 \\ B=-1 \end{cases}$$

$$= \int \left(\frac{1}{t-2} - \frac{1}{t-1} \right) dt = \log|t-2| - \log|t-1| + c =$$

$$= \log \frac{|t-2|}{|t-1|} + c = \log \frac{|e^x-2|}{|e^x-1|} + c$$

$$e^x = t$$

$$\int \frac{\operatorname{senh} x}{\cosh x + 1} \, dx = \int \frac{\frac{e^x - e^{-x}}{2}}{\frac{e^x + e^{-x}}{2} + 1} \, dx =$$

$$= \int \frac{e^x - e^{-x}}{e^x + e^{-x} + 2} \, dx$$

pongo $e^x = t \qquad x = \log t \qquad dx = \frac{1}{t} \, dt$

$$e^{-x} = \frac{1}{e^x} = \frac{1}{t} = t^{-1}$$