Lezione 26: integrali

T.E. 1-03-11

$$\int_1^3 \ln^2 x \, dx$$

$$\int \ln^2 x \, dx \underset{pp.}{=} x \ln^2 x - \underbrace{\int \cancel{x} \cdot \frac{2 \ln x}{\cancel{x}} \, dx}_{(*)} =$$

$$1 \cdot \ln^2 x \rightarrow \quad 1 = D(x)$$

$$D(\ln^2 x) = 2 \frac{\ln x}{x}$$

$$(*) = \int \ln x \, dx \underset{pp.}{=} x \ln x - \int \underbrace{\cancel{x} \cdot \frac{1}{\cancel{x}}}_{\int 1 \, dx = x} dx = x \ln x - x + c_1$$

$$\int \ln^2 x \, dx = x \ln^2 x - 2\left(x \ln x - x \right) + c =$$

$$= x \ln^2 x - 2x \ln x + 2x + c$$

Integrale definito:

$$\int_1^2 \ln^2 x \, dx = x \ln^2 x - 2x \ln x + 2x \Big|_1^2 =$$

sostituendo

$$= 2 \ln^2 2 - 2 \cdot 2 \ln 2 + 2 \cdot 2 - 1 \underbrace{\ln^2 1}_{0} + 2 \cdot 1 \cdot \underbrace{\ln 1}_{0} - 2 \cdot 1 =$$

$$= 2 \ln^2 2 - 4 \ln 2 + 4 - 2 =$$

$$= 2 \ln^2 2 - 4 \ln 2 + 2$$

T.E. $\displaystyle\int_1^2 \frac{\operatorname{arctg} x}{x^2}\,dx =$ (11-02-04 n°6)

$$\int \operatorname{arctg} x \cdot \frac{1}{x^2}\,dx \underset{\text{p.p.}}{=} -\frac{1}{x}\operatorname{arctg} x - \int -\frac{1}{x}\cdot\frac{1}{1+x^2}\,dx =$$

$$D(\operatorname{arctg} x) = \frac{1}{1+x^2}$$

$$\frac{1}{x^2} = D\left(-\frac{1}{x}\right)$$

$$= -\frac{1}{x}\operatorname{arctg} x + \underbrace{\int \frac{1}{x(1+x^2)}\,dx}_{(*)\ \text{Funzioni Fratte}}$$

$$\frac{1}{x(1+x^2)} = \frac{A}{x} + \frac{Bx+C}{1+x^2} = \frac{A(1+x^2)+x(Bx+C)}{x(1+x^2)}$$

$$x^3(A+B) + Cx + A = 1 \qquad \text{identità polinomiale}$$

$$\begin{cases} A+B=0 \\ C=0 \\ A=1 \end{cases} \qquad \begin{cases} A=1 \\ B=-A=-1 \\ C=0 \end{cases}$$

$2x = D(1+x^2)$

$$(*) \int \frac{1}{x(1+x^2)}\,dx = \int \left(\frac{1}{x} - \frac{2x}{2(1+x^2)}\right)dx = \ln|x| - \frac{1}{2}\ln(1+x^2) + C_1$$

$$\int \frac{\operatorname{arctg} x}{x^2}\,dx = -\frac{1}{x}\operatorname{arctg} x + \ln|x| - \frac{1}{2}\ln(1+x^2) + C$$

$$\int_1^2 f(x)\,dx = -\frac{1}{x}\operatorname{arctg} x + \ln|x| - \frac{1}{2}\ln(1+x^2)\Big|_1^2 =$$

$$= -\frac{1}{2}\operatorname{arctg} 2 + \ln 2 - \frac{1}{2}\ln(1+4) + \underbrace{\operatorname{arctg} 1}_{\pi/4} - \underbrace{\cancel{\ln|1|}}_{0} + \frac{1}{2}\ln 2 =$$

$$= -\frac{\operatorname{arctg} 2}{2} + \frac{\pi}{4} + \frac{3}{2}\ln 2 - \frac{1}{2}\ln(5) = -\frac{\operatorname{arctg} 2}{2} + \frac{\pi}{4} + \ln\frac{2}{5} + \ln\sqrt{2}$$

T.E. 4/07/11 n°6

$\mathcal{F} :]0,+\infty[\to \mathbb{R}$ primitiva di $f :]0,+\infty[\to \mathbb{R}$

$f(x) = \dfrac{3}{x^4+x^2}$; $\mathcal{F}(1) = -3 \Rightarrow \lim\limits_{x\to+\infty} \mathcal{F}(x) = ?$

$$\int \frac{3}{x^4+x^2}\,dx = \int \frac{3}{x^2(x^2+1)}\,dx$$

$$\frac{3}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1} = \frac{Ax(x^2+1)+B(x^2+1)+x^2(Cx+D)}{x^2(x^2+1)} =$$

$$\underline{Ax^3} + Ax + \underbrace{Bx^2} + B + \underline{Cx^3} + \underbrace{Dx^2} = 3$$

$$\begin{cases} A+C=0 \\ B+D=0 \\ A=0 \\ B=3 \end{cases} \Rightarrow \begin{cases} A=0 \\ B=3 \\ C=-A=0 \\ D=-B=-3 \end{cases}$$

$$\int \frac{3}{x^4+x^2}\,dx = \int \left[\frac{3}{x^2} - \frac{3}{x^2+1}\right] dx = -\frac{3}{x} - 3\operatorname{arctg} x + C$$

$$f(x) = -\frac{3}{x} - 3\operatorname{arctg} x + C$$

$$f(1) = -3$$

$$C + \cancel{\left(-\frac{3}{1}\right)} - 3\operatorname{arctg} 1 = \cancel{-3}$$

$$C = 3\operatorname{arctg} 1 = 3\frac{\pi}{4} = \frac{3}{4}\pi$$

$$f(x) = -\frac{3}{x} - 3\operatorname{arctg} x + \frac{3}{4}\pi$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}\left(-\underset{\frac{3}{\infty}}{\frac{3}{x}} - \underset{3\frac{\pi}{2}}{3\operatorname{arctg} x} + \frac{3}{4}\pi\right) = -\frac{3}{2}\pi + \frac{3}{4}\pi = -\frac{3}{4}\pi$$

T.E. (20.01.11) n°6

$\mathcal{F}$ primitiva di $f(x) = \dfrac{e^{3x} - 2e^{x}}{1+e^{2x}}$

$\mathcal{F}(0) = 1 \Rightarrow \mathcal{F}(1) = ?$

$$\int \frac{e^{3x} - 2e^{x}}{1+e^{2x}}\,dx = \int \frac{t^{\cancel{3}\,2} - 2\cancel{t}}{1+t^2} \cdot \frac{1}{\cancel{t}}\,dt =$$

$e^x = t$

$x = \ln t$

$dx = \frac{1}{t}\,dt$

$$= \int \frac{t^2 - 2}{1+t^2}\,dt = \int \frac{(t^2+1)-1-2}{1+t^2}\,dt = \int \left(\frac{t^2+1}{t^2+1} - \frac{3}{1+t^2}\right)dt =$$

$$= \int \left(1 - \frac{3}{1+t^2}\right)dt = t - 3\,\mathrm{arctg}\,t + c$$

$$= t - 3 \operatorname{arctg} t + c$$

$$f(x) = e^x - 3 \operatorname{arctg} e^x + C$$

$$f(0) = 1$$

$$\underset{1}{\cancel{e^0}} - 3 \underbrace{\operatorname{arctg} \underset{1}{e^0}}_{\pi/4} + C = \cancel{1} \Rightarrow C = \frac{3}{4}\pi$$

$$f(x) = e^x - 3 \operatorname{arctg} e^x + \frac{3}{4}\pi$$

$$f(1) = e - 3 \operatorname{arctg} e + \frac{3}{4}\pi$$

Equazioni differenziali

es 1:

$$\int a(x)\,dx = A(x)$$

$$\begin{cases} u'(x) + x^2 u(x) = \underbrace{4x^2}_{b(s)} \\ u(0) = 5 \end{cases}$$

1° ordine
coeff. continui

calcolo $A(x) = \int_0^x s^2\,ds = \frac{s^3}{3}\Big|_0^x = \frac{x^3}{3}$

$$\int_0^x e^{A(s)}\, b(s)\, ds = \int_0^x 4 s^2 e^{\frac{s^3}{3}}\, ds =$$

$d\left(\frac{s^3}{3}\right) = s^2$

$$s^2 = D\left(\frac{s^3}{3}\right)$$

$$= 4\, e^{\frac{s^3}{3}}\Big|_0^x = 4\, e^{\frac{x^3}{3}} - 4$$

$$u(x) = e^{-A(x)}\left(u(0) + \int_0^x e^{A(s)} b(s)\, ds\right) = e^{-\frac{x^3}{3}}\left(5 + 4\, e^{\frac{x^3}{3}} - 4\right) =$$

$$= 4 + e^{-\frac{x^3}{3}}$$

Es 3

$$\begin{cases} x^3 u'(x) + x^2 u(x) = 6 & \forall x > 0 \\ u(1) = 0 \end{cases}$$

Per poter applicare la formula risolutiva l'equazione deve essere ridotta in forma normale:

$$\frac{x^3}{x^3} u'(x) + \frac{x^2}{x^3} u(x) = \frac{6}{x^3} \qquad \forall x > 0$$

Ridotta in forma normale

$$u'(x) + \underbrace{\frac{1}{x}}_{a(x)} u(x) = \underbrace{\frac{6}{x^3}}_{b(x)}$$

$$A(x) = \int_1^x \frac{1}{s}\, ds = \ln(x)$$

$| \; \forall x > 0 \Rightarrow$ non è necessario il $| \; |$

$$\int_1^x e^{A(s)}\, b(s)\, ds = \int_1^x e^{\ln(s)} \cdot \frac{6}{s^3}\, ds =$$

$$= 6 \int_1^x \frac{\cancel{s}}{s^{\cancel{3}2}}\, ds = 6 \int_1^x \frac{1}{s^2}\, ds =$$

$$= 6 \left[-\frac{1}{s} \Big|_1^x \right] = 6 \left(-\frac{1}{x} + 1 \right)$$

Componendo la formula risolutiva

$$u(x) = \underbrace{e^{-\ln x}}_{e^{\ln x^{-1}}} \left(\overset{u(1)=0}{0} + 6 \left(1 - \frac{1}{x} \right) \right) =$$

$$= \frac{6}{x} \left(1 - \frac{1}{x} \right) = \frac{6}{x} - \frac{6}{x^2}$$

es 3

$$\begin{cases} y'(x) + \frac{1}{x^2}\, y(x) = \underbrace{e^{\frac{1}{x}} \operatorname{arctg} x}_{b(x)} & \forall x>0 \\ y(1) = 0 \end{cases}$$

$$A(x) = \int_1^x \frac{1}{s^2}\, ds = -\frac{1}{s}\Big|_1^x = -\frac{1}{x} + 1 = 1 - \frac{1}{x}$$

$$\int_1^x e^{A(s)}\, b(s)\, ds = \int_1^x e^{\left(1-\frac{1}{s}\right)}\, e^{\frac{1}{s}} \operatorname{arctg} s\, ds =$$

$$= \int_1^x e^{1 - \cancel{\frac{1}{s}} + \cancel{\frac{1}{s}}} \operatorname{arctg} s\, ds =$$

$$= e\int_1^x \operatorname{arctg} s\, ds \underset{pp.}{=} e\left(s \operatorname{arctg} s \Big|_1^x - \int_1^x \frac{2s}{2(1+s^2)}\, ds \right) =$$

$$D(\operatorname{arctg} s) = \frac{1}{1+s^2}$$

$$= e\left(x \operatorname{arctg} x - \operatorname{arctg} 1 - \frac{1}{2} \ln (1+x^2) \Big|_{1}^{x} \right) =$$

$$= e\left(x \operatorname{arctg} x - \frac{\pi}{4} - \frac{1}{2} \ln (1+x^2) + \frac{1}{2} \ln 2 \right)$$

Sostituendo nella formula risolutiva

$$y(x) = e^{\frac{1}{x} - 1}\left(0 + e\left(x \operatorname{arctg} x - \frac{\pi}{4} - \frac{1}{2} \ln (1+x^2) + \frac{1}{2} \ln 2 \right)\right) =$$

$$= e^{\frac{1}{x}}\left(x \operatorname{arctg} x - \frac{1}{2} \ln (1+x^2) + \ln \sqrt{2} - \frac{\pi}{4} \right)$$

es 4

$$\begin{cases} y'(x) + \frac{2x}{4+x^2}\, y(x) = x^{\frac{1}{5}} \\ y(0) = 1 \end{cases}$$

$$A(x) = \int_0^x \frac{2s}{4+s^2}\, ds = \ln(4+s^2)\Big|_0^x = \ln(4+x^2) - \ln 4$$

$$\int_0^x e^{\ln\left(\frac{4+s^2}{4}\right)} s^{\frac{1}{5}}\, ds = \int_0^x \frac{4+s^2}{4} \cdot s^{\frac{1}{5}}\, ds =$$

$$= \int_0^x \left(s^{\frac{1}{5}} + \frac{1}{4} s^{\frac{11}{5}}\right) ds = \frac{5}{6} s^{6/5} + \frac{5}{64} s^{\frac{16}{5}} \Big|_0^x =$$

$$= \frac{5}{6} x^{6/5} + \frac{5}{64} x^{16/5}$$

$$y(x) = e^{-\ln\frac{4+x^2}{4}}\left(1 + \frac{5}{6}x^{\frac{6}{5}} + \frac{5}{64}x^{\frac{16}{5}}\right) =$$

$$= \left(\frac{4+x^2}{4}\right)^{-1}\left(1 + \frac{5}{6}x^{6/5} + \frac{5}{64}x^{\frac{16}{5}}\right) =$$

$$= \frac{4}{4+x^2}\left(1 + \frac{5}{6}x^{6/5} + \frac{5}{64}x^{16/5}\right)$$

Buon lavoro a voi!