Lezione n° 19: derivazione & C.

1. Si consideri la funzione

$$y = 2 \cdot e^{-4x} - e^{3x}$$

determinare $k \in \mathbb{R}$: $\boxed{y''' + 2y' - 8y = k\, e^{3x}}$

$$y = f(x) = 2 \cdot e^{-4x} - e^{3x}$$

$$y' = 2\, e^{-4x}(-4) - e^{3x} \cdot 3 = -8\, e^{-4x} - 3\, e^{3x}$$

$$y'' = -8\, e^{-4x}(-4) - 3\, e^{3x} \cdot 3 = 32\, e^{-4x} - 9\, e^{3x}$$

$$y''' = -128\, e^{-4x} - 27\, e^{3x}$$

$$\underbrace{-128\, e^{-4x} - 27 e^{3x}}_{y'''} + 2\,\big(\underbrace{-8 e^{-4x} - 3 e^{3x}}_{y'}\big) - 8\,\big(\underbrace{2e^{-4x} - e^{3x}}_{y}\big) = k\, e^{3x}$$

$$(-128 - 16 - 16)\, e^{-4x} + (-27 - 6 + 8)\, e^{3x} = k\, e^{3x}$$

$$(-160)\, e^{-4x} - 25 e^{3x} = k e^{3x}$$

$$k = \frac{-160\, e^{-4x} - 25 e^{3x}}{e^{3x}} = \quad \in \mathbb{R}\,?$$

$$= -160\, e^{-4x-3x} - 25 \implies k \text{ è variabile}$$

variabile

no val reale

no costante

se fosse $y'' + 2y' - 8y = k e^{3x}$

$$\underbrace{32 e^{-4x} - 9 e^{3x}}_{y''} + 2\underbrace{\left(-8 e^{-4x} - 3 e^{3x}\right)}_{y'} - 8\underbrace{\left(2e^{-4x} - e^{3x}\right)}_{y} = k e^{3x}$$

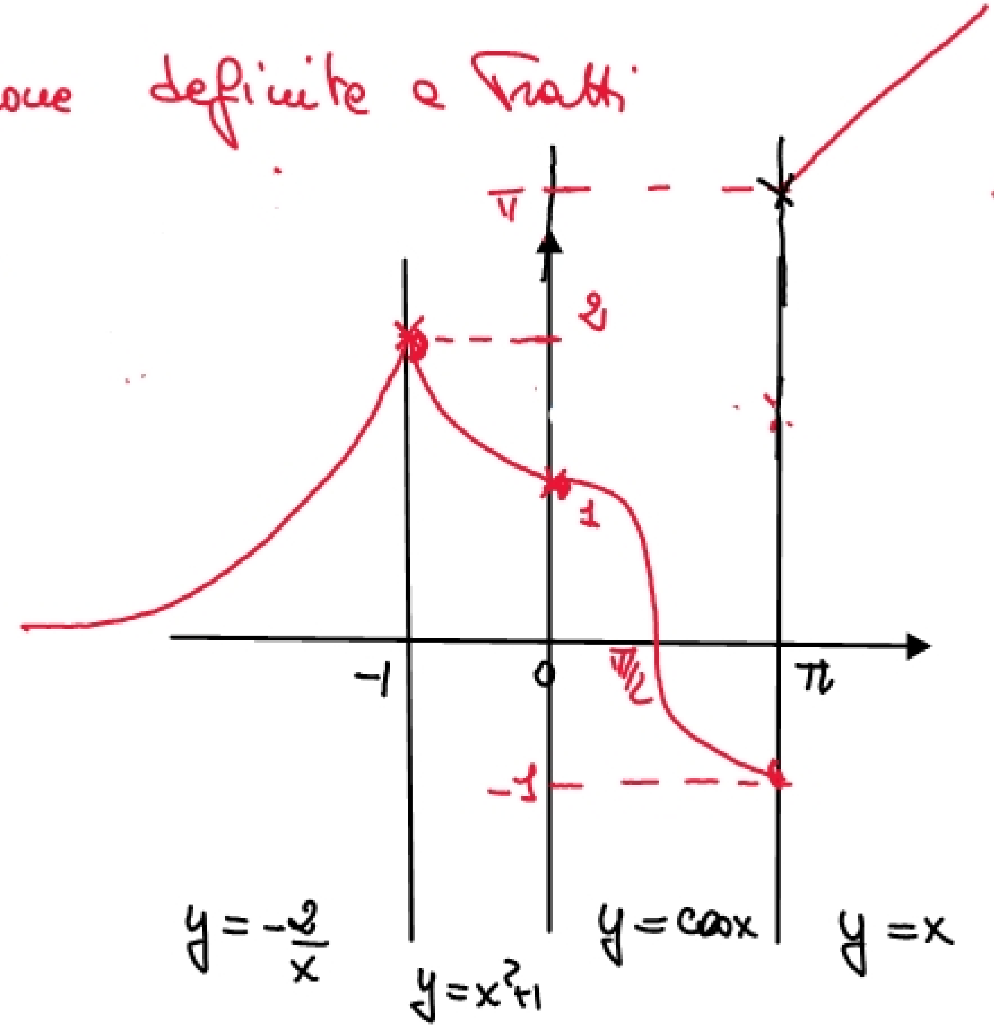
$$e^{-4x}\underbrace{(32 - 16 - 16)}_{0} + e^{3x}(-9 - 6 + 8) = k e^{3x}$$

$$+ e^{\cancel{3x}}(-7) = k e^{\cancel{3x}}$$

$K = -7$ k è un valore reale costante

- Considerare la seguente funzione definita a tratti

$$f: x \to \begin{cases} -2/x & x < -1 \\ x^2+1 & -1 \leq x < 0 \\ \cos x & 0 \leq x \leq \pi \\ x & x > \pi \end{cases}$$



$$\lim_{x \to -1^-} -\frac{2}{x} = 2$$

$$f(-1) = (-1)^2+1 = 2$$

$$\lim_{x \to 0^-} (x^2+1) = 1$$

$$f(0) = \cos 0 = 1$$

$$f(\pi) = \cos \pi = -1$$

$$\lim_{x \to \pi^+} x = \pi$$

La funzione è continua su $\mathbb{R}$ (sul suo dominio) ad eccezione del punto $x = \pi$, discontinuità a salto.

derivabilità

$x = -1$

$$\lim_{\substack{x \to -1^+ \\ x_0 = -1}} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to -1} \frac{-\frac{2}{x} - 2}{x + 1} =$$

$$= \lim_{x \to -1} \frac{-2 - 2x}{x(x+1)} = \lim_{x \to -1} \frac{-2\cancel{(x+1)}}{x\cancel{(x+1)}} = 2$$

$y'_+ = 2x \qquad m_+(-1) = -2$

rette Tangenti

$(-1, 2)$ sinistra $\qquad y - 2 = 2(x+1) \to y = 2x + 4$

destra $\qquad y - 2 = -2(x+1) \to y = -2x$

$x=0$

$$\lim_{x\to 0^-} \frac{\overbrace{x^2+1}^{f(0)}-1}{x-0} = \lim_{x\to 0^-} \frac{x^{\cancel{2}}}{\cancel{x}} = 0$$

$$m_+(0) = -\operatorname{sen}(0) = 0$$

$$m_-(0) = m_+(0)$$ tangente orizzontale $y=1$

$x=\pi$ non studio la derivabilità perché la funzione non è continua in $x=\pi$

la $f(x)$ ✓ è continua in $R \setminus \{\pi\}$

✓ è derivabile in $R \setminus \{-1, \pi\}$

✓ $(-1,2)$ è un punto angoloso e $(0,1)$ a tangente orizz.

Calcolo derivate

$$D\left(\arcsin \sqrt{x-4}\right) =$$

$$= \underbrace{\frac{1}{\sqrt{1-\left(\sqrt{x-4}\right)^2}}}_{D(\arcsin t)} \cdot \underbrace{\frac{1}{2\sqrt{x-4}}}_{D\left(\sqrt{x-4}\right)} \underbrace{\frac{D(x-4)}{1}} =$$

$$= \frac{1}{2} \frac{1}{\sqrt{1-x+4}} \frac{1}{\sqrt{x-4}} =$$

$$= \frac{1}{2} \frac{1}{\sqrt{5-x}} \cdot \frac{1}{\sqrt{x-4}} = \ldots$$

$$D(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$$

$$D(\arccos x) = -\frac{1}{\sqrt{1-x^2}}$$

$$D(\operatorname{arctg} x) = \frac{1}{1+x^2}$$

$$D(\operatorname{arcctg} x) = -\frac{1}{1+x^2}$$

N.B.

D [funzione inversa]

$$\left(f^{-1}\right)'(y_0) = \frac{1}{f'(x_0)}$$

$y = \arcsen \sqrt{x-4}$

$\sqrt{\ }$ $\left\{ \begin{array}{l} x-4 \geq 0 \\ -1 \leq \text{Arg} \leq 1 \end{array} \right.$ arcsen

$\left\{ \begin{array}{l} x \geq 4 \\ -1 \leq \sqrt{x-4} \leq 1 \end{array} \right.$

$\sqrt{x-4} \geq 0$

$\left\{ \begin{array}{l} \sqrt{x-4} \geq -1 \quad \forall x \\ \sqrt{x-4} \leq 1 \end{array} \right.$

$\downarrow$

$x-4 \leq 1$

$x \leq 5$

dom f : $4 \leq x \leq 5$

dom f' : $\left\{ \begin{array}{l} 5-x > 0 \\ x-4 > 0 \end{array} \right.$ $\left\{ \begin{array}{l} x < 5 \\ x > 4 \end{array} \right.$ $4 < x < 5$

$$D\left(\operatorname{arctg}\frac{x+1}{x-1}\right) = \qquad \text{dom} f: \quad x \neq 1$$

$$= \underbrace{\frac{1}{1+\left(\frac{x+1}{x-1}\right)^2}}_{D(\operatorname{arctg} t)} \cdot \underbrace{\frac{\cancel{x}-1-\cancel{x}-1}{(x-1)^2}}_{D(\text{argomento})} = \frac{-2}{\frac{(x-1)^2+(x+1)^2}{\cancel{(x-1)^2}} \cdot \cancel{(x-1)^2}} =$$

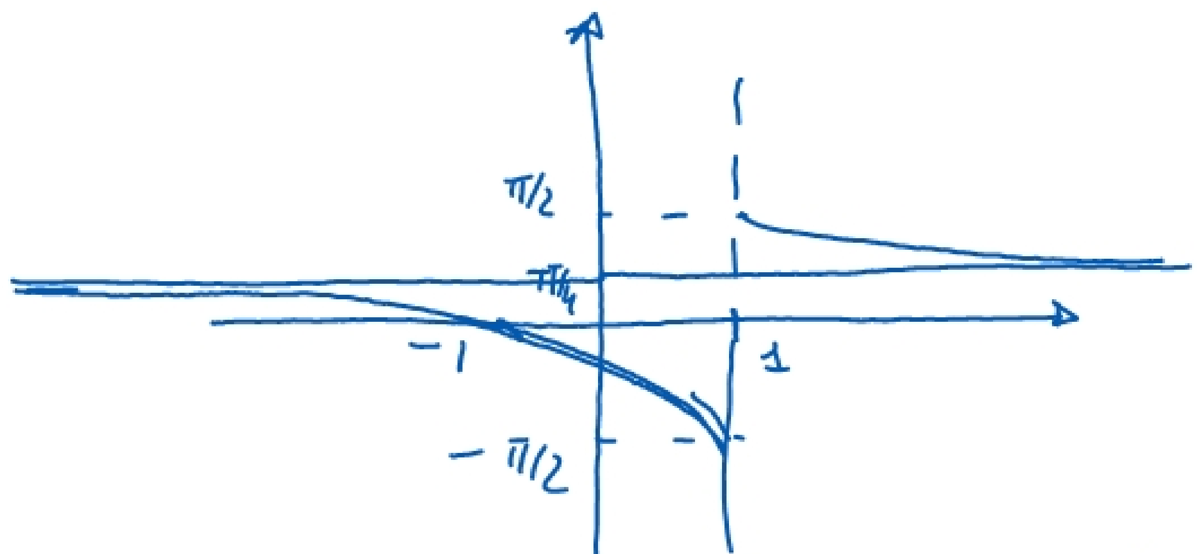
$$= \frac{-2}{2x^2+2} = \frac{-1}{x^2+1} \; \underline{<0} \;\; \forall x$$

per $x=1$? la $f(x)$ non è definita

$$\lim_{x \to 1} \operatorname{arctg}\frac{x+1}{x-1} = \operatorname{arctg}(\infty) \; ?$$

$$\lim_{x\to 1^-} \operatorname{arctg}\left(\frac{x+1}{x-1}\right) = \operatorname{arctg}(-\infty) = -\frac{\pi}{2}$$

$$\lim_{x\to +1^+} \operatorname{arctg}\left(\frac{x+1}{x-1}\right) = \operatorname{arctg}(+\infty) = \pi/2$$



$$\lim_{x\to +\infty} \operatorname{arctg}\left(\frac{x+1}{x-1}\right) = \operatorname{arctg} 1 = \pi/4 \quad \Big\} \quad \lim_{x\to -\infty} \operatorname{arctg}\left(\frac{x+1}{x-1}\right) = \operatorname{arctg}(1) = \pi/4$$

$$D\left(\arccos(3x+1)\right) = \underbrace{-\frac{1}{\sqrt{1-(3x+1)^2}}}_{D(\arccos)} \cdot \underbrace{3}_{D(\text{argom})} = \frac{-3}{\sqrt{1-9x^2-6x-1}} =$$

$$= \frac{-3}{\sqrt{-9x^2-6x}} < 0$$

dom f

$$-1 \le 3x+1 \le 1$$
$$-2 \le 3x \le 0$$
$$-\tfrac{2}{3} \le x \le 0$$

dom f'

$$-9x^2-6x > 0$$
$$3x^2+2x < 0$$
$$-\tfrac{2}{3} < x < 0$$

(estremi esclusi)

$$D\left(\operatorname{arcctg}\left(\frac{x}{2}\right)\right) = \underbrace{\frac{-1}{1+\frac{x^2}{4}}}_{D\,\operatorname{arcctg}} \quad \underbrace{\frac{1}{2}}_{D(\arg)} =$$

$$= -\frac{\cancel{4}^{\,2}}{\cancel{2}(4+x^2)} = -\frac{2}{4+x^2} < 0$$

dom f $\quad \forall x \in \mathbb{R}$

dom f' $\quad \forall x \in \mathbb{R}$

TEO DI DE L'HOSPITAL

f, g funzioni reali continue in $[a,b]$ $\qquad \frac{0}{0}$

derivabili in $]a,b[$

escluso al più $x_0 \in]a,b[$

con $g'(x) \neq 0 \quad \forall x \in]a,b[\; ; \; x \neq x_0$

$f(x_0) = 0 \qquad g(x_0) = 0$

se esiste finito $\lim_{x \to x_0} \frac{f'(x)}{g'(x)} \Rightarrow$ esiste $\lim_{x \to x_0} \frac{f(x)}{g(x)}$

$$\lim_{x \to x_0} \frac{f'(x)}{g'(x)} = \lim_{x \to x_0} \frac{f(x)}{g(x)}$$

$$\lim_{x\to 0} \frac{e^{x} - e^{\operatorname{sen} x}}{x + \operatorname{sen} x} = \frac{1-1}{0+0} = \frac{0}{0} \quad \text{F. I.}$$

$$\underset{H}{=} \lim_{x\to 0} \frac{e^{x} - e^{\operatorname{sen} x}\cos x}{1+\cos x} = \frac{1 - 1\cdot 1}{1+1} = 0$$

$$\lim_{x\to 0} \frac{\operatorname{arctg} x}{x} \underset{H}{=} \lim_{x\to 0} \frac{1}{(1+x^2)\cdot 1} = 1$$

$$\lim_{x\to \frac{\pi}{4}} \frac{\log \operatorname{Tg} x}{\operatorname{ctg} 2x} \underset{H}{=} \lim_{x\to \frac{\pi}{4}} \left\{ \frac{1}{\operatorname{Tg} x} (1+\operatorname{Tg}^2 x) \cdot \frac{1}{(-1-\operatorname{ctg}^2 2x)\cdot 2} \right\} =$$

$$= \frac{1}{1} (\cancel{1+1}) \frac{1}{(-1-0)\cancel{2}} = -1$$

$$\lim_{x \to 0} \frac{\log \operatorname{sen} x}{\operatorname{ctg} x} = \frac{-\infty}{\pm\infty} = \frac{\infty}{\infty} \quad F.I.$$

$$\underset{H}{=} \lim_{x \to 0} \frac{\frac{1}{\operatorname{sen} x} \cdot \cos x}{-\frac{1}{\operatorname{sen}^2 x}} =$$

$$= \lim_{x \to 0} \frac{\cos x}{\cancel{\operatorname{sen} x}} \cdot \left(- \operatorname{sen}^2 x\right) = 0$$