Lezione 11 - successioni

FORMULA DI DEMOIVRE - STIRLING

$$n! = n^n \cdot e^{-n} \sqrt{2\pi n} \; e^{\frac{\vartheta_n}{12n}} \qquad \underline{0 < \vartheta_n < 1}$$

$$\lim_{n \to +\infty} \sqrt[n]{\frac{n!}{n^n}} = \lim_{n \to +\infty} \left(\frac{n!}{n^n}\right)^{1/n} =$$

$$= \lim_n \sqrt[n]{\frac{\cancel{n^n} \cdot e^{-n} \sqrt{2\pi n} \; e^{\vartheta_n / 12n}}{\cancel{n^n}}} =$$

$$= \frac{\vartheta_n}{12 n^2} \to 0 \Rightarrow e^0 = 1$$

$$= \lim_n e^{-1} \sqrt[2n]{2\pi n} \; \underbrace{e^{\frac{\vartheta_n}{12 n \cdot n}}}_{\to 1} = \qquad \log(2\pi n)^{1/2n}$$

$$\simeq \lim_n \frac{1}{e} (2\pi n)^{\frac{1}{2n}} = \lim_n \frac{1}{e} \quad e$$

$$= \lim_{n} \frac{1}{e} \, e^{\frac{\log(2\pi n)}{2n}} =$$

$$= \frac{1}{e} \cdot \underbrace{e^{0}}_{1} = \frac{1}{e}$$

$$\Rightarrow \lim_{n \to +\infty} \sqrt[n]{\frac{n!}{n^n}} = \frac{1}{e}$$

$$\lim_{n\to+\infty} \frac{(2e)^n \sqrt{n^3+1}\; n!}{(2n)^n\, n^2} =$$

$$(\pm 2)^2 = 4$$

$$\sqrt{4} = |2|$$

$$\underset{+}{\sqrt{4}} = \underset{\pm}{\pm 2} \quad ?$$

$$= \lim_{n} \frac{\cancel{2^n}\, e^n\, \cancel{n} \overbrace{\sqrt{1+\underbrace{\tfrac{1}{n^2}}_{\to 0}}}^{\to 1}\; n!}{\cancel{2^n}\, n^n\, n^{\cancel{2}}} \simeq$$

$$\simeq \lim_{n} \frac{e^n\, n!}{n^n\, n} =$$

$$= \lim_{n} \frac{\cancel{e^n}\, \cancel{n^n}\, \cancel{e^{-n}} \sqrt{2\pi n}\; e^{\overbrace{\frac{\vartheta_n}{12n}}^{\to 0}}}{\cancel{n^n}\; n} =$$

with $\overbrace{\hphantom{xx}}^{1}$ over $e^{\frac{\vartheta_n}{12n}}$

$$= \lim_{n} \frac{\sqrt{2\pi}\; n^{1/2}}{n} = 0$$

(prevale il denominatore!)

TEMA D'ESAME (29.01.10)

$$\lim_{n\to+\infty} \frac{\sqrt[4]{n}\,\left(\cos\frac{1}{n^4}-1\right)}{\sqrt{\log\left(1+\frac{5}{n^7}\right)+n}-\sqrt{n}} = \frac{\infty\cdot 0}{\infty-\infty} \quad \text{F. I.}$$

$$= \lim_{n\to+\infty} \sqrt[4]{n}\,\underbrace{\left(\cos\frac{1}{n^4}-1\right)\frac{\cos\frac{1}{n^4}+1}{\cos\frac{1}{n^4}+1}}_{-\operatorname{sen}^2\frac{1}{n^4}\ (*)}\;\frac{\sqrt{\log\left(1+\frac{5}{n^7}\right)+n}+\sqrt{n}}{\left(\sqrt{\log\left(1+\frac{5}{n^7}\right)+n}-\sqrt{n}\right)\left(\sqrt{\log\left(1+\frac{5}{n^7}\right)+n}+\sqrt{n}\right)}$$

$$= \lim_{n\to+\infty} \sqrt[4]{n}\,\frac{\left(\cos^2\frac{1}{n^4}-1\right)}{\underbrace{\underbrace{\cos\frac{1}{n^4}}_{\simeq 1}+1}_{2}}\;\frac{\sqrt{\log\left(1+\frac{5}{n^7}\right)+n}+\sqrt{n}}{\log\left(1+\frac{5}{n^7}\right)+\cancel{n}-\cancel{n}} =$$

$$(*)\quad -\operatorname{sen}^2\frac{1}{n^4} = -\frac{\operatorname{sen}\frac{1}{n^4}\cdot\operatorname{sen}\frac{1}{n^4}}{\frac{1}{n^4}\cdot\frac{1}{n^4}\cdot n^8} = -\frac{\operatorname{sen}^2\frac{1}{n^4}}{\frac{1}{n^8}\cdot n^8}$$

attenzione :

$$\lim_{n \to +\infty} n \operatorname{sen} \frac{1}{n} = 1$$

$$\lim_{n \to +\infty} \frac{\operatorname{sen} \frac{1}{n}}{\frac{1}{n}} = 1 \qquad \lim_{n \to +\infty} \frac{\operatorname{sen} \frac{1}{n^2}}{\frac{1}{n^2}} = 1$$

$$\left(\lim_{t \to 0} \frac{\operatorname{sen} t}{t} = 1 \right) \qquad t = \frac{1}{n}$$

$$\lim_{n \to +\infty} \operatorname{sen}^2 \frac{1}{n} \cdot \log n = \lim_{n \to +\infty} \frac{\operatorname{sen} \frac{1}{n} \cdot \operatorname{sen} \frac{1}{n} \cdot \log n}{n \cdot \frac{1}{n}}$$

$$= \lim_{n \to +\infty} \overbrace{\frac{\operatorname{sen} \frac{1}{n}}{n \cdot \frac{1}{n}} \cdot \frac{\operatorname{sen} \frac{1}{n}}{n \cdot \frac{1}{n}}}^{\operatorname{sen}^2 \frac{1}{n}} \cdot \log n = \lim_{n \to +\infty} \frac{\log n}{n^2} = 0$$

$$\frac{1}{n} \cdot \frac{1}{n} = \frac{1}{n^2}$$

$$\lim_{n\to+\infty} n \operatorname{sen}\frac{1}{n^2} =$$

$$= \lim_{n\to+\infty} n \frac{n}{n} \operatorname{sen}\frac{1}{n^2} =$$

$$= \lim_{n\to+\infty} \underbrace{\left(n^2 \operatorname{sen}\frac{1}{n^2}\right)}_{1} \frac{1}{n} = 0$$

$$= \lim_{n\to+\infty} \frac{\sqrt[4]{n}}{\cancel{\infty}} \left(- \frac{\overbrace{\operatorname{sen}^2 \frac{1}{n^4}}^{\to 1}}{\underbrace{\frac{1}{n^8}}_{\frac{1}{n^4}\cdot\frac{1}{n^4}} \cdot n^8}\right)$$

$$\frac{\sqrt{n}\left(\sqrt{\frac{\log\left(1+\frac{5}{n^7}\right)}{n}+1}+1\right)}{\underbrace{\frac{n^7}{5}\log\left(1+\frac{5}{n^7}\right)}_{n^7/5}} =$$

$$\frac{n^7}{5}\log\left(1+\frac{5}{n^7}\right) \to 1 \qquad \log\left(1+\frac{5}{n^7}\right) \to 0$$

$$\lim_{n\to+\infty} \log\left(1+\frac{5}{n^7}\right) = \frac{\frac{n^7}{5}\log\left(1+\frac{5}{n^7}\right)^{n^7/5}}{n^7/5}$$

$$\left(1+\frac{5}{n^7}\right)^{n^7/5} \rightsquigarrow e$$

$$\log\left(1+\frac{5}{n^7}\right) = \frac{\frac{n^7}{5}\log\left(1+\frac{5}{n^7}\right)}{n^7/5} = \frac{\log\left(1+\frac{5}{n^7}\right)^{n^7/5}}{n^7/5}$$

$$\sqrt{\lim_{n\to+\infty} \frac{\frac{n^7}{5}\log\left(1+\frac{5}{n^7}\right)}{\frac{n^7}{5}\,n}} = \sqrt{\lim_{n\to+\infty} \frac{\log\left(1+\frac{5}{n^7}\right)^{n^7/5}}{n^8/5}} = \frac{5\cdot 1}{\infty} = 0$$

$\rightsquigarrow \log e$

$$= \lim_{n \to +\infty} \sqrt[4]{n} \left(- \frac{1}{n^8} \right) \cdot \frac{\sqrt{n}}{\frac{1}{n^7/5}} =$$

$$= \lim_{n \to +\infty} \left[\frac{n^{\frac{1}{4}} \cdot n^{1/2} \cdot n^{7}}{n^8 \quad 5} \right] =$$

$$= -\frac{1}{5} \lim_{n \to +\infty} \frac{n^{7+\frac{3}{4}}}{n^8} = 0$$

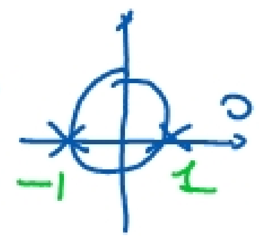
prevale il denominatore !!!

Insiemi e successioni

• $A = \{ \underbrace{2\cos(n\pi)}_{b_n} + \underbrace{\operatorname{sen}\left(2^{-n}\frac{\pi}{2}\right)}_{c_n}, n \in \mathbb{N} \}$

$a_n = b_n + c_n$

$$b_n = 2\cos(n\pi) = \begin{cases} 2 & n = 2m \text{ pari} \\ -2 & n = 2m+1 \text{ dispari} \end{cases}$$

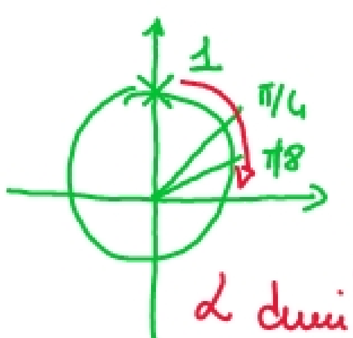


$$c_n = \operatorname{sen}\left(2^{-n}\frac{\pi}{2}\right) = \operatorname{sen}\left(\frac{1}{\underbrace{2^n}_{\to\infty}} \cdot \frac{\pi}{2}\right)$$

$\underbrace{\qquad\qquad}_{\to 0}$

$c_0 = \operatorname{sen}\left(2^0 \frac{\pi}{2}\right) = \operatorname{sen}\frac{\pi}{2} = 1$

$c_1 = \operatorname{sen}\left(2^{-1}\frac{\pi}{2}\right) = \operatorname{sen}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$



α diminuisce $\Rightarrow$ $\operatorname{sen}(\alpha) \searrow$

$$\lim_{n\to+\infty} \operatorname{sen}\left(\frac{1}{2^n}\cdot\frac{\pi}{2}\right) = \operatorname{sen} 0 = 0$$

c_n è una successione a termini positivi, decrescente

$$0 < c_n \leq 1 \qquad \inf\{c_n\} = 0$$

$$\max\{c_n\} = 1$$

$$a_n = \begin{cases} 2 + \underbrace{\operatorname{sen}\left(\frac{\pi}{2^{n+1}}\right)}_{\text{decrescente}} & n = 2m \text{ pari} \\ -2 + \operatorname{sen}\left(\frac{\pi}{2^{n+1}}\right) & n = 2m+1 \text{ dispari} \end{cases}$$

$\sup A = \max A = 3$

$\inf A = -2$

$a_0 = b_0 + c_0 = 2 + 1 = 3$

$a_1 = b_1 + c_1 = -2 + \frac{\sqrt{2}}{2} < 0$