Lezione 13: serie

Serie geometrica

$$\sum_{n=0}^{\infty} q^n \Rightarrow \begin{cases} |q| > 1 & \text{divergente} \\ q = 1 & \text{divergente} \\ |q| < 1 & \text{CONVERGENTE} \\ |q| = -1 & \text{indeterminata} \end{cases}$$

q = ragione della serie geometrica

$|q| > 1$ $\qquad \sum_{i=0}^{n} q^i \qquad S_n = \dfrac{1-q^{n-1}}{1-q}$

$|q| < 1$ $\qquad S_n = \dfrac{1}{1-q}$

• determinare la somma delle seguenti serie :

$$\sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n \qquad -1 < q = \frac{1}{4} < 1 \qquad S = \frac{1}{1-q} = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$$

$n = 0, \ldots$

$$\sum_{n=1}^{\infty} \left(\frac{3}{5}\right)^n = \qquad -1 < q = \frac{3}{5} < 1$$

$\boxed{\text{Operatore lineare } \Sigma}$

$n = 1, \ldots$

$$= \sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^n - 1 \qquad S = \frac{1}{1-q} - 1 = \frac{1}{1-\frac{3}{5}} - 1 = \frac{5}{2} - 1 = 3/2$$

$$\sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^{n+1} = \sum_{n=0}^{\infty} \left[\left(\frac{1}{4}\right)^n \cdot \underbrace{\left(\frac{1}{4}\right)^1}_{\text{cost.}}\right] = \frac{1}{4} \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n$$

$$S = \frac{1}{4} \cdot \frac{4}{3} = 1/3$$

$$\sum_{n=3}^{\infty}\left(\frac{3}{7}\right)^n = \sum_{n=0}^{\infty}\left(\frac{3}{7}\right)^n - \underbrace{\left(\frac{3}{7}\right)^0}_{a_0} - \underbrace{\left(\frac{3}{7}\right)^1}_{a_1} - \underbrace{\left(\frac{3}{7}\right)^2}_{a_2} =$$

$-1 < q = \frac{3}{7} < 1$

$n = 3 \,-\,-\,-$

$$= \sum_{n=0}^{\infty}\left(\frac{3}{7}\right)^n - 1 - \frac{3}{7} - \frac{9}{49}$$

$$S = \frac{1}{1-\frac{3}{7}} - 1 - \frac{3}{7} - \frac{9}{49} = \frac{7}{4} - \frac{10}{7} - \frac{9}{49} =$$

$$= \frac{343-280-36}{196} = \frac{27}{196}$$

• Determinare per quali valori di $x \in \mathbb{R}$ le seguenti serie sono convergenti:

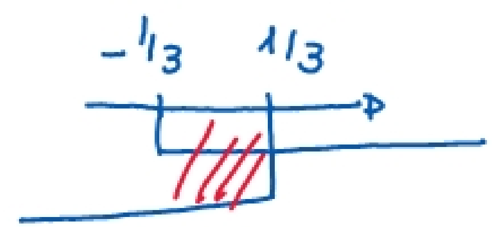
$$\sum_{n=0}^{+\infty} (3x)^n$$

Convergenza come serie geometrica se

$$|q| < 1$$

$$|3x| < 1 \Rightarrow -1 < 3x < 1 \Rightarrow -\frac{1}{3} < x < \frac{1}{3}$$

$$\Rightarrow \begin{cases} 3x > -1 \\ 3x < 1 \end{cases}$$



$$\sum_{n=0}^{+\infty}\left(\frac{x^2-x}{x^2-4}\right)^n$$ converge come serie geometrica se

$$|q|<1$$

$$-1<\frac{x^2-x}{x^2-4}<1 \Leftarrow \left|\frac{x^2-x}{x^2-4}\right|<1$$

$$\begin{cases} \dfrac{x^2-x}{x^2-4} > -1 & (1) \\[2mm] \dfrac{x^2-x}{x^2-4} < 1 & (2) \end{cases}$$

(2) $$\frac{\cancel{x^2}-x-\cancel{x^2}+4}{x^2-4}<0$$

$$\frac{-x+4}{x^2-4}<0 \Rightarrow -\frac{x-4}{(x-2)(x+2)}<0$$

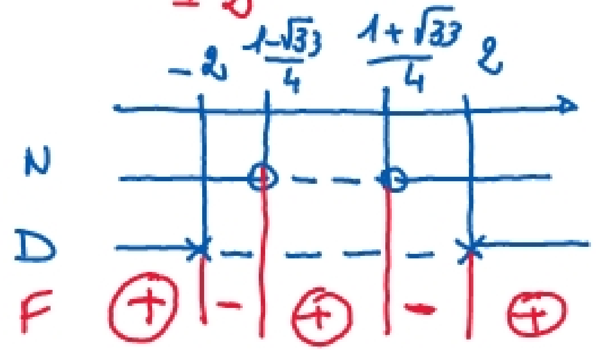
$$\frac{x-4}{(x-2)(x+2)}>0$$

(1) $$\frac{x^2-x+x^2-4}{x^2-4}>0$$

$$\frac{2x^2-x-4}{x^2-4}>0 \qquad x_{1,2}=\frac{1\pm\sqrt{1+32}}{4}$$

$\sqrt{33}$ $\sim 5,\ldots$

± 2

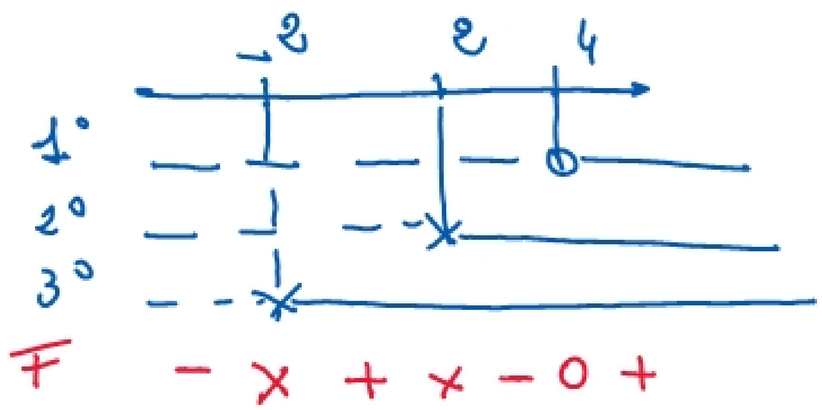


$x<-2$

$\frac{1-\sqrt{33}}{4}<x<\frac{1+\sqrt{33}}{4}$

$x>2$

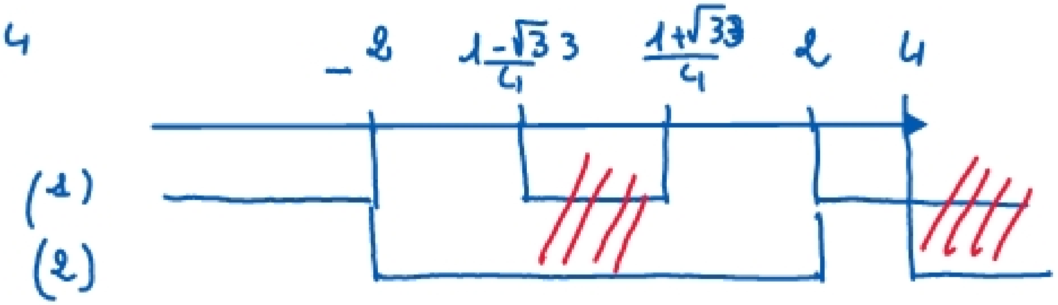
$$\frac{x-4}{(x-2)(x+2)} > 0$$



$-2 < x < 2$

$x > 4$

$$\begin{cases} (1) & x < -2 \ \cup \ \frac{1-\sqrt{33}}{4} < x < \frac{1+\sqrt{33}}{4} \ \cup \ x > 2 \\ (2) & -2 < x < 2 \ \cup \ x > 4 \end{cases}$$



$$\frac{1-\sqrt{33}}{4} < x < \frac{1+\sqrt{33}}{4} \ \cup \ x > 4$$

$$\sum_{n=1}^{\infty} \frac{2^n}{5^n+1} = ? \quad \text{convergente.}$$

$\sum b_n$ confrontabile con $\sum_{n=1}^{\infty} \frac{2^n}{5^n+1}$

$$\frac{2^n}{5^n+1} < \frac{2^n}{5^n} = \left(\frac{2}{5}\right)^n$$

$$\sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^n \quad \text{serie geometrica}$$

$$q = \frac{2}{5}$$

$$|\text{ragione}| < 1$$

$$\Downarrow$$

$$\sum_{n=1}^{\infty} \frac{2^n}{5^n+1} \text{ conv.} \Leftarrow \text{Converge}$$

(criterio del confronto)

$$\sum_{n=1}^{+\infty} \frac{2^n}{5^n+1}$$

criterio del confronto asintotico del rapporto

$$\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to+\infty} \underbrace{\frac{2^{n+1}}{5^{n+1}+1}}_{a_{n+1}} \cdot \underbrace{\frac{5^n+1}{2^n}}_{1/a_n} =$$

$$= \lim_{n\to+\infty} \frac{\cancel{2^n}\cdot 2}{\underbrace{5^{n+1}}_{\cancel{5^n}\cdot 5}\Big(1+\underbrace{\frac{1}{5^{n+1}}}_{0}\Big)} \cdot \frac{\cancel{5^n}\Big(1+\overbrace{\frac{1}{5^n}}^{n\to 0}\Big)}{\cancel{2^n}} = 2/5 < 1$$

$L < 1 \Rightarrow$ convergenza

serie utili per il confronto

serie armonica generalizzata
$$\sum_{n=1}^{+\infty} \frac{1}{n^\alpha} \begin{cases} \text{se } \alpha > 1 & \text{converge} \\ \text{se } \alpha \leq 1 & \text{diverge} \end{cases}$$

serie geometrica
$$\sum_{n=0}^{+\infty} q^n \begin{cases} |q| < 1 & \text{converge} \\ |q| \geq 1 & \text{diverge} \end{cases}$$

serie
$$\sum_{n=2}^{+\infty} \frac{1}{n\,(\log n)^\alpha} \begin{cases} \alpha > 1 & \text{convergente} \\ \alpha \leq 1 & \text{divergente} \end{cases}$$

$$\sum_{n=2}^{+\infty} \frac{1}{n^\alpha \log n} \begin{cases} \alpha > 1 & \text{convergente} \\ \alpha \leq 1 & \text{divergente} \end{cases}$$

- $$\sum_{n=1}^{+\infty} \frac{1}{(n!)^2} \qquad a_n = \frac{1}{(n!)^2}$$

$$\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to+\infty} \frac{1}{\left((n+1)!\right)^2} \cdot (n!)^2 =$$

$$= \lim_{n\to+\infty} \frac{1}{\left((n+1)\, n!\right)^2} \cdot (n!)^2 =$$

$$= \lim_{n\to+\infty} \frac{1}{(n+1)^2 \cancel{(n!)^2}} \cdot \cancel{(n!)^2} =$$

$$= \lim_{n\to+\infty} \frac{1}{(n+1)^2} = 0$$

converge

tema d'esame [1. 02. 05]

$$\sum_{n=1}^{+\infty} \frac{3^n \, n!}{n^n}$$

diverge positivamente

$L > 1$

criterio del rapporto (asintotico)

$$\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to+\infty} \underbrace{\frac{3^{n+1}(n+1)!}{(n+1)^{n+1}}}_{a_{n+1}} \cdot \underbrace{\frac{n^n}{3^n \, n!}}_{1/a_n} =$$

$$= \lim_{n\to+\infty} \frac{\cancel{3^n}\cdot 3 \, \overset{\to 1}{(n+1)} \, \cancel{n!}}{\underbrace{n^{n+1}}_{\cancel{n^n}\cdot n}\left(1+\frac{1}{n}\right)^{n+1}} \cdot \frac{\cancel{n^n}}{\cancel{3^n}\,\cancel{n!}} = \frac{3}{e} > 1$$

$$\lim_{n\to+\infty} \frac{n+1}{n} = 1 \quad \Big| \quad \lim_{n\to+\infty} \overset{\to e}{\left(1+\frac{1}{n}\right)^n} \cdot \overset{\to 1}{\left(1+\frac{1}{n}\right)^1}$$

$$\sum_{n=1}^{+\infty} \frac{(n+1)!\, n^n + 2^n (n!)^2}{2^n\, n!\, n^n} =$$

convergente

$(n+1)\,\cancel{n!}$

$$= \sum_{n=1}^{+\infty} \frac{(n+1)!\,\cancel{n^n}}{2^n\,\cancel{n!}\,\cancel{n^n}} + \sum_{n=1}^{+\infty} \frac{\cancel{2^n}\,(n!)^{\cancel{2}}}{\cancel{2^n}\,\cancel{n!}\,n^n} =$$

$$= \sum_{n=1}^{+\infty} \underbrace{\frac{n+1}{2^n}}_{a_n} + \sum_{n=1}^{+\infty} \underbrace{\frac{n!}{n^n}}_{b_n}$$

$$\lim_{n\to+\infty} \frac{(n+1)+1}{2^{n+1}} \cdot \frac{2^n}{n+1} =$$

$$= \lim_{n\to+\infty} \frac{n+2}{\cancel{2^n}\cdot 2} \cdot \frac{\cancel{2^n}}{n+1} = \frac{1}{2} < 1$$

$\downarrow 1$

CONV.

$$\lim_{n\to+\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} =$$

$$= \lim_{n\to+\infty} \frac{(n+1)\,\cancel{n!}}{\cancel{n^n}\cdot n\,\underbrace{\left(1+\frac{1}{n}\right)^n}_{e}\underbrace{\left(1+\frac{1}{n}\right)}_{1}} \cdot \frac{\cancel{n^n}}{\cancel{n!}} =$$

$\to 1$

$$= \frac{1}{e} < 1$$

Conv.

$$\sum_{n=1}^{+\infty} \left(\frac{11}{10}\right)^n \cdot \frac{1}{n^5}$$

serie geometrica

$|q| > 1$

div.

serie armonica generalizzata

$\alpha > 1$

convergente

$$\lim_{n\to+\infty} \left(\frac{11}{10}\right)^{n+1} \cdot \frac{1}{(n+1)^5} \cdot \frac{n^5}{\left(\frac{11}{10}\right)^n} =$$

$$= \lim_{n\to+\infty} \underbrace{\frac{n^5}{(n+1)^5}}_{\frac{\cancel{n^5}}{\cancel{n^5}\left(1+\frac{1}{n}\right)^5}} \cdot \frac{\cancel{\left(\frac{11}{10}\right)^n} \cdot \frac{11}{10}}{\cancel{\left(\frac{11}{10}\right)^n}} = \frac{11}{10} > 1$$

$\left(1+\frac{1}{n}\right) \to 1$

divergente

$$\sum_{n=1}^{+\infty} \left(\frac{n}{3n-1}\right)^{2n-1}$$

criterio della radice asintotico

$$\lim_{n\to+\infty} \sqrt[n]{\left(\frac{n}{3n-1}\right)^{2n-1}} =$$

$$\frac{2n-1}{n} = 2 - \frac{1}{n}$$

$$= \lim_{n\to+\infty} \left(\frac{n}{3n-1}\right) =$$

$$= \lim_{n\to+\infty} \left(\frac{3n-1}{n}\right)^{-2+\frac{1}{n}} =$$

$$= \lim_{n\to+\infty} \underbrace{\left(3-\frac{1}{n}\right)^{\frac{1}{n}\to 0}}_{\substack{\downarrow \\ 3 \\ 3^0 = 1}} \underbrace{\left(3-\frac{1}{n}\right)}_{\substack{\downarrow \\ 0}}^{-2} = \frac{1}{9} < 1$$

convergente