Lezione 6 : numeri complessi

1) Calcolare il modulo del seguente numero complesso

$z = (2+i)^2 - (1-2i)^2 =$ $\qquad z = x + iy$

$= 4 + \underbrace{i^2}_{} + 4i - (1 + \underbrace{4i^2}_{-4} - 4i) =$ $\qquad i^2 = -1$ $\qquad x, y \in \mathbb{R}$

$= 4 - 1 + 4i - 1 + 4 + 4i =$ $\qquad |z| = \sqrt{x^2 + y^2}$

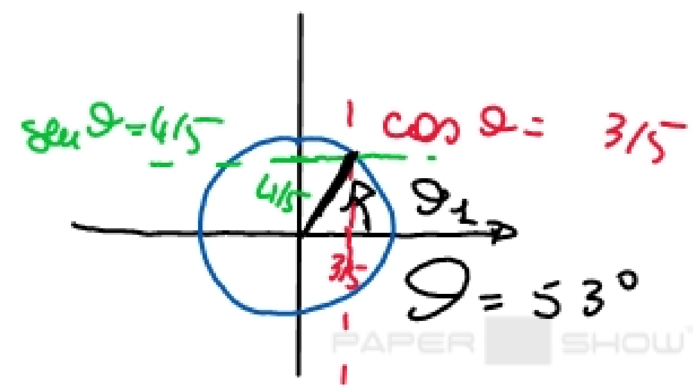
$= 6 + 8i$ $\quad \left[= 10\, e^{53^\circ i} \right.$ $\longrightarrow$ $53^\circ = \frac{53}{180}\pi$

$x = \operatorname{Re} z = 6$ $\qquad |z| = \sqrt{6^2 + 8^2} = \sqrt{36 + 64} = \sqrt{100} =$

$y = \operatorname{Im} z = 8$ $\qquad = 10$

$\rho = 10$

$$\theta = \begin{cases} \cos\theta = \dfrac{\operatorname{Re} z}{\rho} = \dfrac{6}{10} = 3/5 \\ \sin\theta = \dfrac{\operatorname{Im} z}{\rho} = \dfrac{8}{10} = 4/5 \end{cases}$$



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$$z = 6+8i = 10\, e^{\frac{53}{180}\pi i} = 10\left(\cos\frac{53}{180}\pi + i \operatorname{sen}\frac{53}{180}\pi\right)$$

notazione algebrica

notazione esponenziale

notazione trigono-metrica

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2) Verificare che:

$$\frac{(2+i)^2+(1-i)^2}{1-\frac{3}{2}i} = 2i$$

$$\frac{(2+i)^2+(1-i)^2}{1-\frac{3}{2}i}\cdot\frac{1+\frac{3}{2}i}{1+\frac{3}{2}i} =$$

$$= \frac{(4 \overset{+i^2}{-1} + 4i \cancel{+1} \overset{+i^2}{\cancel{-1}} - 2i)\,(1+\frac{3}{2}i)}{1+\frac{9}{4}} =$$

$$= \frac{4}{13}(3+2i)\left(1+\frac{3}{2}i\right) =$$

$$= \frac{\cancel{4}^{2}}{13}\cdot\frac{1}{\cancel{2}_{1}}(3+2i)(2+3i) =$$

$$= \frac{2}{13}(\cancel{6}+4i+9i\overset{+6i^2}{-\cancel{6}}) = \frac{2}{13}\,13i = 2i \quad \text{cvd}$$

Prodotto notevole

$$(a+b)(a-b) =$$
$$= a^2-b^2$$

$$(a+ib)(a-ib)$$
$$= a^2-\underbrace{i^2}_{-1}b^2 =$$
$$= a^2+b^2$$

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3) determinare se esistono valori reali della x per i quali il numero complesso $\dfrac{x-1+3i}{x-2i}$ risulti reale

$$z = \frac{x-1+3i}{x-2i} = \qquad \exists x : \quad z \in \mathbb{R} \Rightarrow \operatorname{Im} z = 0$$

$$= \frac{x-1+3i}{x-2i} \cdot \frac{x+2i}{x+2i} = \frac{[(x-1)+3i]\,(x+2i)}{x^2+4} =$$

$-4i^2$

$$= \frac{x(x-1) + 3xi + 2(x-1)i - 6}{x^2+4} =$$

$$= \frac{(x(x-1)-6)}{x^2+4} + \frac{3x+2(x-1)}{x^2+4}\, i$$

$$\operatorname{Im} z = 0 \Rightarrow \frac{3x+2(x-1)}{x^2+4} = 0 \Rightarrow 3x+2x-2=0$$

$$x = 2/5$$

$$x^2+4 \neq 0 \quad \forall x \in \mathbb{R}$$

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Tema d' esame 11/02/11

ē dato l' insieme

$$A = \left\{ z \in \mathbb{C} : Re\left(\frac{z+i}{z-i}\right) < 0 \text{ e } Im(z + i\bar{z}) \geq 0 \right\}$$

il luogo geometrico rappresentato ē ?

posto $z = x + iy$

si devono individuare i due insiemi di valori di z;

$$\begin{cases} Re\left(\frac{z+i}{z-i}\right) < 0 \\ Im(z + i\bar{z}) \geq 0 \end{cases}$$

valgano contemporaneamente (vere)

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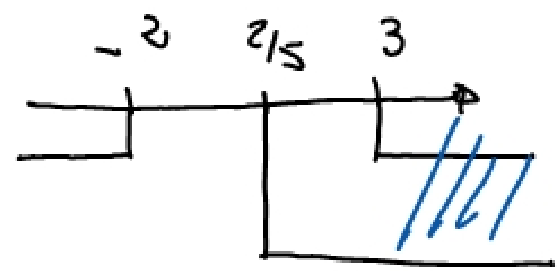
z abbia parte reale e immaginaria strettamente positiva

$\Downarrow$

contemporaneamente vere

$$\begin{cases} \operatorname{Re} z > 0 \\ \operatorname{Im} z > 0 \end{cases} \qquad \begin{cases} \dfrac{x^2 - x - 6}{\underset{+}{x^2+4}} > 0 \\[2ex] \dfrac{5x - 2}{\underset{+}{x^2+4}} > 0 \end{cases} \qquad \begin{cases} x^2 - x - 6 > 0 \\ 5x - 2 > 0 \end{cases}$$

$$\begin{cases} (x-3)(x+2) > 0 \\ x > 2/5 \end{cases} \qquad \begin{cases} x < -2 \cup x > 3 \\ x > 2/5 \end{cases}$$



$X > 3$

$$\frac{z+i}{z-i} = \frac{x+iy+i}{x+iy-i} = \frac{x+i(y+1)}{x+i(y-1)} \cdot \frac{x-i(y-1)}{x-i(y-1)} =$$

$$= \frac{x^2 - xi(y-1) + xi(y+1) + (y+1)(y-1)}{x^2+(y-1)^2} =$$

$$= \frac{x^2+(y^2-1)}{x^2+(y-1)^2} + \frac{xi\,(\cancel{y}+1-\cancel{y}+1)}{x^2+(y-1)^2} =$$

1^e cond. $\quad \frac{x^2+y^2-1}{x^2+(y-1)^2} < 0$

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$$z + iz = x+iy + i(x+iy) =$$

$$= x+iy+ix-y =$$

$$= (x-y) + i\underbrace{(x+y)}_{\mathrm{Im}(\ \ ) \geqslant 0}$$

2e cond. $\quad x+y \geqslant 0$

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$$\begin{cases} \dfrac{x^2+y^2-1}{x^2+(y-1)^2} < 0 \\ x+y \geq 0 \end{cases}$$

$x^2+(y-1)^2 > 0 \quad \forall (x,y) \in R^2$

$[x^2+(y-1)^2 \neq 0 \Rightarrow (x,y) \neq (0,1)]$

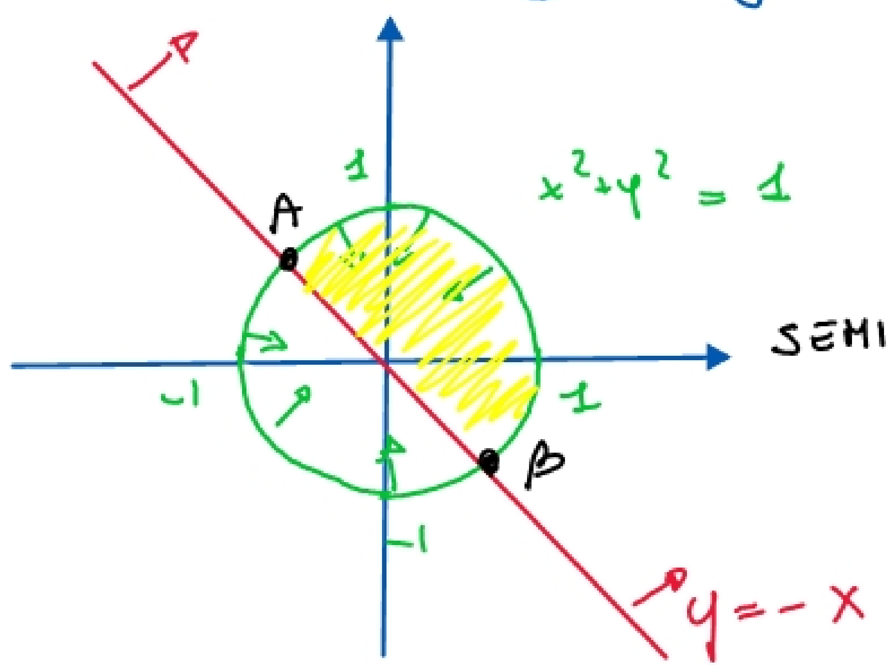
$$\begin{cases} x^2+y^2-1 < 0 \\ x+y \geq 0 \end{cases} \Rightarrow$$

① $x^2+y^2-1=0$ $\qquad x^2+y^2=1$ cfr:

$C(0,0)$

$R=1$

② $x+y=0$ $\qquad y=-x$ bisettrice 2°-4° quad.



$x^2+y^2=1$ $\qquad x^2+y^2-1<0$

SEMICERCHIO

$y \geq -x$

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4) Determinare i numeri complessi soddisfacenti alla condizione

$$(x+iy)^2 = -\frac{1}{2}+\frac{\sqrt{3}}{2}i \qquad \left[z^2 = -\frac{1}{2}+\frac{\sqrt{3}}{2}i\right]$$

$$x^2+2xyi-y^2 = -\frac{1}{2}+\frac{\sqrt{3}}{2}i$$

$$\underbrace{(x^2-y^2)}_{Re} + \overbrace{2xy}^{Im}\, i = -\frac{1}{2}+\frac{\sqrt{3}}{2}i$$

[Identità polinomiale]

$$\Rightarrow \begin{cases} x^2-y^2 = -\frac{1}{2} \\ 2xy = \frac{\sqrt{3}}{2} \end{cases} \qquad (x\neq 0) \quad (y \neq 0) \qquad \begin{cases} x^2-\frac{3}{16x^2} = -\frac{1}{2} \\ y = \frac{\sqrt{3}}{4x} \end{cases}$$

$$16x^4-3+8x^2=0 \qquad 16x^4+8x^2-3=0 \qquad x^2=t$$

$$16t^2+8t-3=0$$

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$$16t^2 + 8t - 3 = 0$$

$$t_{1,2} = \frac{-4 \pm \sqrt{16+48}}{16} = \frac{-4 \pm \sqrt{64}}{16} = \frac{-4 \pm 8}{16} \begin{cases} -\frac{12}{16} & \text{N.A} \\ +\frac{4}{16} = \frac{1}{4} \end{cases}$$

$$\left(\cancel{t}^2 = -\frac{12}{16} \quad \text{n.a.} \right)$$

$$\cancel{t}^2 = \frac{1}{4} \qquad \cancel{t}_1 = +\frac{1}{2} \qquad \cancel{t}_2 = -\frac{1}{2}$$

$$\begin{cases} x_1 = \frac{1}{2} \\ y_1 = \frac{\sqrt{3}}{4x_1} = \frac{\sqrt{3}}{2} \end{cases} \qquad \begin{cases} x_2 = -\frac{1}{2} \\ y_2 = \frac{\sqrt{3}}{4x_2} = -\frac{\sqrt{3}}{2} \end{cases}$$

$$z_1 = \left(\frac{1}{2} ; \sqrt{3}/2 \right) = \frac{1}{2} + \frac{\sqrt{3}}{2} i \qquad z_2 = \left(-\frac{1}{2} ; -\sqrt{3}/2 \right) = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$$

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