# Lezione 13: serie

Serie telescopiche:

$$\sum_{n=0}^{+\infty} a_n = \sum_{n=0}^{+\infty} \left( b_n - b_{n+1} \right)$$

$$a_n = b_n - b_{n+1}$$

$$S_n = \sum_{k=0}^{n} a_k = \sum_{k=0}^{n} \left( b_k - b_{k+1} \right) =$$

$$= \underbrace{b_0 - b_1}_{k=0} + \underbrace{b_1 - b_2}_{k=1} + \underbrace{b_2 - b_3}_{k=2} + \dots +$$

$$+ \underbrace{b_{n-1} - b_n}_{k=n-1} + \underbrace{b_n - b_{n+1}}_{k=n} = \underline{b_0 - b_{n+1}}$$

$$\sum_{n=0}^{\infty} = \lim_{n \to +\infty} S_n = \lim_{n \to +\infty} (b_0 - b_{n+1}) =$$

$$= \lim_{n \to +\infty} b_0 - \lim_{n \to +\infty} b_{n+1} =$$

$$= b_0 - \lim_{n \to +\infty} b_{n+1}$$

$$\sum_{n=1}^{\infty} \left( \frac{1}{n!} - \frac{1}{(n+1)!} \right) = \quad \longleftarrow \left[ \frac{(n+1)! - n!}{n!\,(n+1)!} \right]$$

$$b_n = \frac{1}{n!} \qquad b_{n+1} = \frac{1}{(n+1)!}$$

serie telescopica

$$= b_1 - \lim_{n} b_{n+1} = 1 - \underbrace{\lim_{n \to +\infty} \frac{1}{(n+1)!}}_{=0} = 1$$

La somma di questa serie telescopica vale 1

$$\bullet \quad \sum_{n=1}^{\infty} \left[\log\left(1+\frac{1}{n}\right)\right] =$$

$$= \sum_{n=1}^{\infty} \left[\log\left(\frac{n+1}{n}\right)\right] = \sum_{n=1}^{\infty} \Big(\underbrace{\log(n+1)}_{b_{n+1}} - \underbrace{\log(n)}_{b_n}\Big) =$$

$$= -\left\{ \sum_{n=1}^{\infty} \big(\log n - \log(n+1)\big) \right\} =$$

serie telescopica

$$= -\left\{ b_1 - \lim_{n\to+\infty} \big(\log(n+1)\big) \right\} =$$

$$= -\Big\{ \underbrace{\log 1}_{0} - \underbrace{\lim_{n\to+\infty} \big(\log(n+1)\big)}_{+\infty} \Big\} = +\infty$$

la serie telescopica diverge positivamente

$$\sum_{n=3}^{+\infty} \frac{(n+1)\sin\left(n\frac{\pi}{2}\right) - n\cos\left(n\frac{\pi}{2}\right)}{n\,(n+1)} =$$

$$= \sum_{n=3}^{+\infty} \left[ \frac{\cancel{(n+1)}\sin\left(n\frac{\pi}{2}\right)}{n\,\cancel{(n+1)}} - \frac{\cancel{n}\cos\left(n\frac{\pi}{2}\right)}{\cancel{n}\,(n+1)} \right] =$$

$$\sin\left((n+1)\frac{\pi}{2}\right)$$

$$= \sum_{n=3}^{+\infty} \left[ \frac{\sin\left(n\frac{\pi}{2}\right)}{n} - \frac{\boxed{\cos\left(n\frac{\pi}{2}\right)}}{n+1} \right] =$$

$$\underline{\sin\left((n+1)\frac{\pi}{2}\right)} = \sin\left(n\frac{\pi}{2} + \frac{\pi}{2}\right) = \sin\left(\alpha + \frac{\pi}{2}\right) =$$

$$= \cos(\alpha) = \underline{\cos\left(n\frac{\pi}{2}\right)}$$

$$= \sum_{n=3}^{+\infty} \left( \underbrace{\frac{\sin\left(n\frac{\pi}{2}\right)}{n}}_{b_n} - \underbrace{\frac{\sin\left((n+1)\frac{\pi}{2}\right)}{n+1}}_{b_{n+1}} \right) =$$

$$= b_3 - \lim_{n\to+\infty} b_{n+1} =$$

$$= \frac{\sin\left(3\frac{\pi}{2}\right)}{3} - \lim_{n\to+\infty} \frac{\overbrace{\sin\left((n+1)\frac{\pi}{2}\right)}^{\to 0}}{n+1} \;(*) =$$

$$-1 \le \sin(n+1)\frac{\pi}{2} \le 1$$

$$0 \leftarrow -\frac{1}{n+1} \le \frac{\sin\left((n+1)\frac{\pi}{2}\right)}{n+1} \le \frac{1}{n+1} \to 0$$

$$= -\frac{1}{3}$$

$$*\left( \lim_{n\to+\infty} \frac{\sin(n+1)\frac{\pi}{2}}{n+1} = \lim_{n\to+\infty} \frac{\sin\left(n\frac{\pi}{2}\right)}{n} \right)$$

$$\bullet \quad \sum_{n=1}^{+\infty} \left( \frac{n^3}{e^n} + \frac{n!}{2^n+1} \right) =$$

N.B. 2 serie a termini positivi

$$= \sum_{n=1}^{+\infty} \frac{n^3}{e^n} + \sum_{n=1}^{+\infty} \frac{n!}{2^n+1} \Rightarrow \text{diverge positivamente}$$

studio separatamente le 2 serie :

$$\lim_{n\to+\infty} \frac{(n+1)^3}{e^{n+1}} \cdot \frac{e^n}{n^3} = \lim_{n\to+\infty} \underbrace{\left(\frac{n+1}{n}\right)^3}_{1} \cdot \frac{\cancel{e^n}}{\cancel{e^n} \cdot e} = \frac{1}{e} < 1$$

$$\sum_{n=1}^{+\infty} \frac{n^3}{e^n} \quad \text{convergente}$$

$$\lim_{n\to+\infty} \frac{(n+1)!}{2^{n+1}+1} \cdot \frac{2^n+1}{n!} = \lim_{n\to+\infty} \frac{(n+1)\cancel{n!}}{\cancel{n!}} \cdot \frac{\cancel{2^n}\overbrace{\left(1+\frac{1}{2^n}\right)}^{\to 1}}{\underbrace{\cancel{2^n}\cdot 2}_{2^{n+1}} \underbrace{\left(1+\frac{1}{2^{n+1}}\right)}_{1}} = +\infty$$

$$\sum_{n=1}^{+\infty} \frac{n!}{2^n+1} \quad \text{diverge positivamente}$$

$$\bullet \sum_{n=1}^{\infty} \frac{1}{\ln^{n}(n+1)}$$

criterio della radice:

$$\lim_{n \to +\infty} \sqrt[\cancel{n}]{\frac{1}{\ln^{\cancel{n}}(n+1)}} = \lim_{n \to +\infty} \frac{1}{\underbrace{\ln(n+1)}_{\to +\infty}} = 0$$

$\Rightarrow$ la serie è convergente

- Sia $\alpha \in \mathbb{R}$ e

$$\sum_{n=2}^{+\infty} \left( \frac{e^{\alpha n}}{n^2} + \frac{1}{n^{\alpha+3} \log n} \right) \quad \text{converge} \iff \alpha \ldots$$

---

$$\sum_{n=2}^{+\infty} \frac{e^{\alpha n}}{n^2}$$

(C. N.)

$$\lim_{n \to +\infty} \frac{e^{\alpha n}}{n^2} = \begin{cases} +\infty & \alpha > 0 \\ 0 & \alpha < 0 \end{cases}$$

$\Rightarrow$ se $\alpha > 0$, non è verificata la cond. necessaria,

quindi $\sum_{n=2}^{+\infty} \frac{e^{\alpha n}}{n^2}$ diverge

$\Rightarrow$ se $\alpha < 0$ è verificata la cond. necessaria:

ma non ho la certezza della convergenza.

$\alpha < 0$

$$\frac{e^{\alpha n}}{n^2} = \frac{1}{n^2 e^{-\alpha n}} \leq \frac{1}{n^2} \qquad \forall n \geq 2$$

$$\boxed{(-\alpha) > 0} \Rightarrow \underline{e^{-\alpha n} > 1}$$

$\sum_n \frac{1}{n^2}$ serie armonica generalizzata, $\underline{\text{converge}}$

$\Rightarrow$ per il criterio del confronto

$$\Rightarrow \begin{cases} \sum_{n=2}^{+\infty} \frac{e^{\alpha n}}{n^2} \\ \text{con } \alpha < 0 \end{cases} \quad \text{è convergente}$$

$$\sum_{n=2}^{+\infty} \frac{1}{n^{\alpha+3} \log n}$$

una delle serie di base (per il confronto)

essa è convergente $\iff \alpha+3>1$

$\Rightarrow \alpha > -2$

Per cui, complessivamente

$$\begin{cases} \alpha \le 0 \\ \alpha > -2 \end{cases} \Rightarrow -2 < \alpha \le 0$$

- Sia $\alpha \in \mathbb{R}$

$$\sum_{n=0}^{+\infty} \frac{n^2}{e^{6n\alpha}} \quad \text{converge} \iff \alpha \ldots\ldots$$

$a_n > 0 \quad \forall n \Rightarrow$ serie a termini positivi

criterio della radice asintotico

$$\sqrt[n]{a_n} = \frac{\sqrt[n]{n^2}}{\left(e^{6n\alpha}\right)^{\frac{1}{n}}} = \frac{\sqrt[n]{n^2}}{e^{6\alpha}}$$

$$\lim_{n\to+\infty} \sqrt[n]{a_n} = \lim_{n\to+\infty} \frac{\overbrace{\sqrt[n]{n^2}}^{\to 1}}{e^{6\alpha}} =$$

$$= \frac{1}{e^{6\alpha}}$$

Attenzione:

$$\lim_{n\to+\infty} \sqrt[n]{n^2} = \lim_{n\to+\infty} \underbrace{\sqrt[n]{n}}_{1} \cdot \underbrace{\sqrt[n]{n}}_{1} = 1$$

$$L = \frac{1}{e^{6\alpha}}$$

$$\frac{1}{e^{6\alpha}} < 1 \qquad e^{-6\alpha} < e^{0}$$

$$-6\alpha < 0 \implies \alpha > 0$$

$\sum a_n$ converge

$$\frac{1}{e^{6\alpha}} > 1 \implies \alpha < 0$$

$\sum a_n$ diverge

$$\frac{1}{e^{6\alpha}} = 1 \implies \alpha = 0$$

Se criterio è inefficace, non si applica

se $\alpha = 0$ $\displaystyle\sum_{n=0}^{+\infty} \frac{n^2}{\underbrace{e^0}_{1}} = \sum_{n=0}^{+\infty} n^2 = \sum_{n=0}^{+\infty} \frac{1}{n^{-2}}$

serie armonica generalizzata ($\alpha < 1$)

$\alpha = 2$ $$\sum_{n=1}^{+\infty} n^2 \longrightarrow \sum_{n=1}^{+\infty} \frac{1}{n^{-2}}$$

serie armonica generalizzata

che diverge (esponente è minore di 1)

$$\lim_{n \to +\infty} n^2 = +\infty$$ la C.N. non è soddisfatta

quindi è divergente