Lezione 7 - numeri complessi

1) $\underbrace{z_1, z_2 \in \mathbb{C}}$ soluzioni dell'equazione $\underbrace{(z-5)^2 = -i}$

Allora $z_1 \cdot z_2$ vale ----

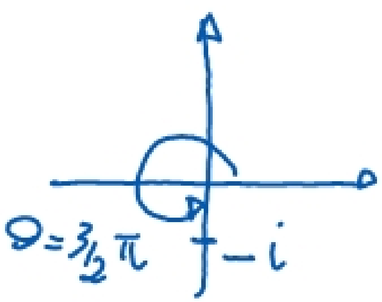
(2° test 11-01-10

$$(z-5)^2 = -i$$

$$z-5 = \pm\sqrt{-i} \longrightarrow z_i = 5 \pm \sqrt{-i}$$

$$z-5 = w \qquad \boxed{w^2 = -i}$$

$$w \Rightarrow \sqrt[2]{-i} \qquad -i = 1\, e^{\frac{3}{2}\pi i}$$



$$\sqrt[2]{-i} = 1\, e^{\frac{\frac{3}{2}\pi + 2k\pi}{2} i}$$

$k = 0, 1$

$$z = 5 + w_i$$

$$w_0 = e^{\frac{3}{4}\pi i} = \left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$$

$$w_1 = e^{\frac{\frac{3}{2}\pi + 2\pi}{2} i} = e^{7/4\,\pi i} = \left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)$$

$$z_1 = 5 + w_0 = 5 + e^{3/4\,\pi i} = 5 + \left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$$

$$z_2 = 5 + w_1 = 5 + e^{7/4\,\pi i} = 5 + \left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)$$

$$z_1 \cdot z_2 =$$

Calcolo delle radici di un numero complesso

$z = a + ib \longrightarrow$ notazione esponenziale $\rightarrow z = \rho e^{i\theta}$

$$\rho = \sqrt{a^2+b^2}$$

$$\theta = \begin{cases} \operatorname{sen}\theta = \dfrac{b}{\sqrt{a^2+b^2}} \\ \cos\theta = \dfrac{a}{\sqrt{a^2+b^2}} \end{cases}$$

$\left[\ \theta = \operatorname{arctg} \dfrac{b}{a}\right.$ ATTENZIONE! $\left.\right]$

$\downarrow$

$$\sqrt[n]{z} = \sqrt[n]{a+ib}$$

$$\left[= \sqrt[n]{\rho}\ e^{i \frac{\theta + 2k\pi}{n}} = \right]$$

$$= \sqrt[n]{\rho}\left(\underbrace{\cos \frac{\theta+2k\pi}{n}}_{a_1} + i \underbrace{\operatorname{sen} \frac{\theta+2k\pi}{n}}_{b_1} \right)$$

(Ritorno in forma algebrica)

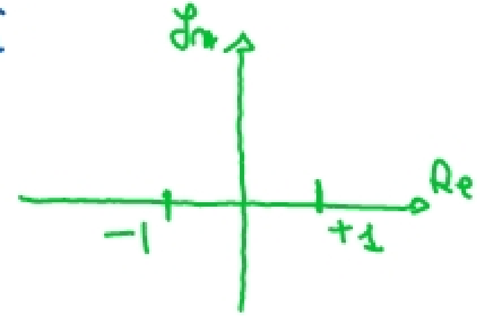
n = radice n-esima

$k = 0, \dots, n-1$

$$z^2 = 1$$

$$z = \pm\sqrt{1} = \pm 1$$

2 sol. reali



$$z^2 = -1 = 1e^{\pi i} \qquad \rho = 1 \quad \vartheta = \pi$$

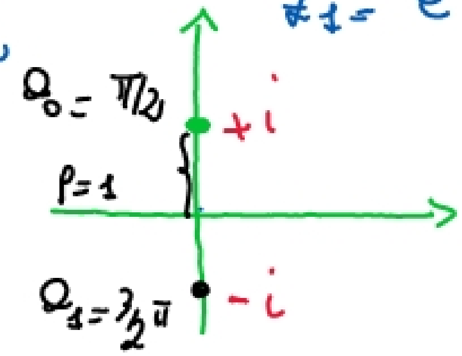
$\vartheta = \pi$

$K = 0, 1$

$n = 2$

$$z_0 = e^{\frac{\pi + 2\cdot 0\pi}{2}} = e^{\pi/2}$$

$$z_1 = e^{\frac{\pi + 2\pi}{2}} = e^{3/2\pi}$$



$z_0 = +i$

$z_1 = -i$

$$z^2 = -1$$
$$z^2 + 1 = 0$$
$$(z-i)(z+i) = 0$$

$$z^4 = 1$$

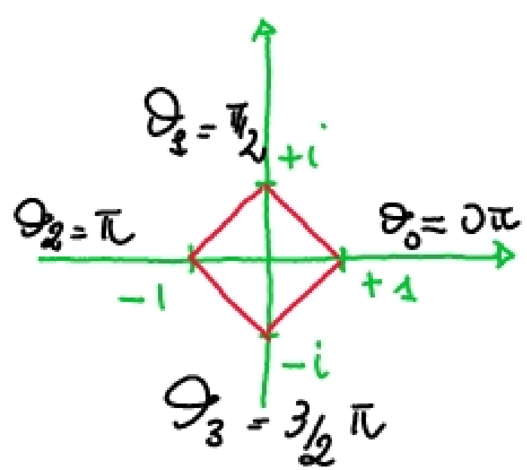
$$z^4 - 1 = 0$$

$$(z^2 - 1)(z^2 + 1) = 0$$

$$(z-1)(z+1)\ (z^2 - i^2) = 0$$

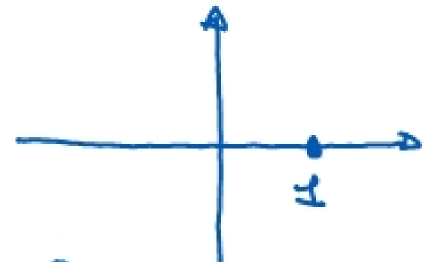
$$(z-1)(z+1)\ (z-i)(z+i) = 0$$

$$z_{1,2} = \pm 1$$
$$z_{3,4} = \pm i$$



$z^4 = +1$

$w = 1$ $\qquad$ $a = 1 \quad (\operatorname{Re} w)$
$\qquad\qquad\qquad$ $b = 0 \quad (\operatorname{Im} w)$



$\rho = 1 \qquad \vartheta = \begin{cases} \operatorname{sen}\vartheta = 0 \\ \cos\vartheta = 1 \end{cases} \Rightarrow \vartheta = 0\pi$

$\sqrt[4]{1}$ $\qquad$ $k = 0, 1, 2, 3$ $\qquad$ $\vartheta_k = \dfrac{\vartheta + 2k\pi}{4}$ $\qquad$ $\sqrt[4]{\rho} = \sqrt[4]{1} = 1$

$n = 4$

$k = 0$

$$z_0 = 1 \cdot \left(\cos \frac{0\pi + 0\pi}{4} + i \operatorname{sen} 0\pi \right) = 1$$

$$z_1 = 1 \cdot \left(\cos \frac{0\pi + 2\pi}{4} + i \operatorname{sen} \pi/2 \right) = i$$

$$z_2 = 1 \left(\cos \frac{0\pi + 4\pi}{4} + i \operatorname{sen} \pi \right) = -1$$

$$z_3 = 1 \left(\cos \frac{0\pi + 6\pi}{4} + i \operatorname{sen} \tfrac{3}{2}\pi \right) = -i$$

$z^4 = -1 = 1\, e^{\pi i}$

$K = 0, 1, 2, 3$

$n = 4$

$\rho = 1$

$\sqrt[4]{\rho} = \sqrt[4]{1} = 1$

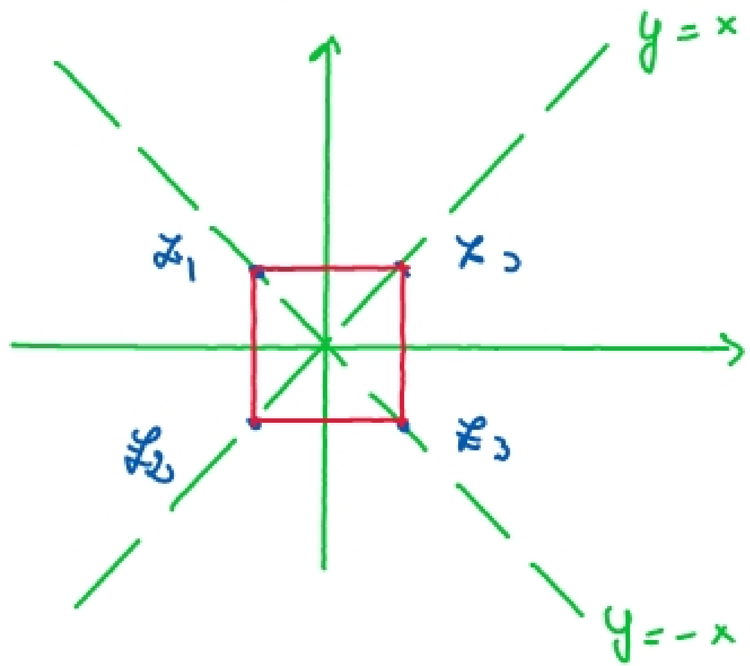
$\theta = \pi$

$z_0 = e^{\frac{\pi + 2\cdot 0 \pi}{4} i} = e^{\pi/4\, i} = \frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2}$

$z_1 = e^{\frac{\pi + 2\cdot 1 \cdot \pi}{4} i} = e^{3/4 \pi i} = -\frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2}$

$z_2 = e^{\frac{\pi + 2\cdot 2 \pi}{4} i} = e^{\frac{5}{4}\pi i} = -\frac{\sqrt{2}}{2} - i \frac{\sqrt{2}}{2}$

$z_3 = e^{\frac{\pi + 2\cdot 3\cdot \pi}{4} i} = e^{7/4 \pi i} = \frac{\sqrt{2}}{2} - i \frac{\sqrt{2}}{2}$



$\rho = 1$

(T. E. 23-10-10)

2) Una delle radici terze complesse di:

$$w = \frac{4}{\sqrt{2}}\left[\left|\underbrace{\frac{\sqrt{2}}{2}}_{a} + i\underbrace{\frac{\sqrt{2}}{2}}_{b}\right| - i\,e^{7\pi i}\right]$$

vale - - - - -

$$w = \frac{4}{\sqrt{2}}\left[\overbrace{\sqrt{\frac{2}{4}+\frac{2}{4}}}^{\sqrt{a^2+b^2}} - i\,\underbrace{e^{\pi i}}\right] = \frac{4}{\sqrt{2}}\left[\underbrace{\sqrt{\frac{4}{4}}}_{1} - i(-1)\right] =$$

$$= \frac{4}{\sqrt{2}}\left[1+i\right] = 4\left[\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = 4\left[\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right] = 4e^{\frac{\pi}{4}i}$$

$$1+i = \sqrt{2}\cdot e^{\pi/4\, i} \longrightarrow = \frac{4}{\sqrt{2}}\cdot\sqrt{2}\,e^{\pi/4\, i} = 4e^{\frac{\pi}{4}i}$$

$$\rho_1 = \sqrt{1+1} = \sqrt{2}$$

$$\vartheta = \begin{cases} \operatorname{sen}\vartheta = \frac{1}{\sqrt{2}} = \sqrt{2}/2 \\ \cos\vartheta = \frac{1}{\sqrt{2}} = \sqrt{2}/2 \end{cases} \qquad \vartheta = \pi/4$$

$$\sqrt[3]{w} \Rightarrow w_i = \sqrt[3]{4}\; e^{\frac{\frac{\pi}{4}+2k\pi}{3}i}$$

$$\sqrt[3]{w} \Rightarrow \quad w_i = \sqrt[3]{4}\; e^{\frac{\frac{\pi}{4}+2k\pi}{3} i} \qquad k = 0, 1, 2$$

$$i = 0, 1, 2$$

$$w_0 = \sqrt[3]{4}\; e^{\pi/12\, i}$$

$$w_1 = \sqrt[3]{4}\; e^{\frac{\frac{\pi}{4}+2\pi}{3} i} = \sqrt[3]{4}\; e^{\frac{3}{4}\pi i} =$$

$$w_2 = \sqrt[3]{4}\; e^{\frac{\frac{\pi}{4}+4\pi}{3} i} = \sqrt[3]{4}\; e^{\frac{17}{12} i}$$

$$\Rightarrow w_1 = \sqrt[3]{4}\left(\cos \frac{3}{4}\pi + i \operatorname{sen} \frac{3}{4}\pi\right) = \sqrt[3]{4}\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$$

(T.E. 21-03-10)

3) L'insieme dei punti $z \in \mathbb{C}$ t.c. $\mathrm{Re}\left(\frac{1}{z}\right) = 2$

$z = x + iy \rightarrow \frac{1}{z} = \frac{1}{x+iy} \cdot \frac{x-iy}{x-iy} = \frac{x-iy}{x^2+y^2} =$

$= \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2} i$

$\left[\frac{1}{z} = \frac{\mathrm{Re}\, z}{\rho^2} - \frac{\mathrm{Im}\, z}{\rho^2} i\right]$

$\mathrm{Re}\left(\frac{1}{z}\right) = 2 \Rightarrow \frac{x}{x^2+y^2} = 2 \qquad (x,y) \neq (0,0)$

$2x^2 + 2y^2 - x = 0$

$x^2 + y^2 - \frac{1}{2}x = 0 \qquad \begin{array}{l} a = -\frac{1}{2} \\ b = 0 \end{array} \rightarrow c\left(\frac{1}{4}, 0\right)$

$R = \sqrt{\frac{1}{16} + 0 + 0} = \frac{1}{4}$

$\mathrm{CFR}_{\setminus(0,0)} \Leftarrow c\left(\frac{1}{4}, 0\right) \quad R = 1/4$

(T.E. 29-06-10)

4) Il luogo dei punti $z \in \mathbb{C}$:

$$z\left(z+\frac{\sqrt{3}}{3}i\right)\ \mathrm{Re}\left(1+2i+z+\sqrt{3}\,i\,\bar{z}\right)=0$$

(legge di annullamento del prodotto)

$z = x+iy$
$\bar{z} = x-iy$

$z = 0 \qquad x+iy=0 \qquad (x,y)=(0,0) \iff \begin{cases} x=0 \\ y=0 \end{cases}$ 1 punto

$z+\frac{\sqrt{3}}{3}i=0 \qquad x+iy=-\frac{\sqrt{3}}{3}i \iff \begin{cases} x=0 \\ y=-\frac{\sqrt{3}}{3} \end{cases} \qquad (x,y)=\left(0,-\frac{\sqrt{3}}{3}\right)$

1 punto

$\mathrm{Re}\left(1+2i+z+\sqrt{3}\,i\,\bar{z}\right)=0$

$$1+2i+x+iy+\sqrt{3}i(x-iy) = \underline{1}+2i+\underline{x}+iy+\sqrt{3}xi+\underline{\sqrt{3}y} =$$
$$=\left(1+x+\sqrt{3}y\right)+\left(2+y+\sqrt{3}x\right)i$$

$1+x+\sqrt{3}y=0$ retta r:

$(0,0)\in r$? $\quad 1\neq 0$ NO

$\left(0,-\frac{\sqrt{3}}{3}\right)\in r$? $\quad \underbrace{1+0-\sqrt{3}\cdot\frac{\sqrt{3}}{3}}_{1}=0$

1 RETTA + 1 PUNTO

(T.E. 6-09-10)

5) Le 3 soluzioni in $\mathbb{C}$ dell'equazione

$$2(z+\bar{z}) - 3\,\mathrm{Im}(z) = z^2 - 3|z|^2$$

$z = x + iy$

$\bar{z} = x - iy$

$z^2 = (x+iy)^2$

$|z| = \sqrt{x^2+y^2}$

$z + \bar{z} = x + \underline{iy} + x - \underline{iy} = 2x$

$z \cdot \bar{z} = (x+iy)(x-iy) = x^2 + y^2$

$z - \bar{z} = \underline{x} + iy \underline{-x} + iy = 2yi$

$$\underbrace{2 \cdot 2x - 3y}_{\text{numero Reale}} = (x+iy)^2 - 3(x^2+y^2)$$

$$\underbrace{4x - 3y}_{\text{reale}} = \underbrace{x^2 - y^2 - 3(x^2+y^2)}_{\text{reale}} + \underbrace{2xyi}_{\text{Imm.}}$$

νέο σύστημ. $\begin{cases} 4x-3y = x^2-y^2-3(x^2+y^2) \\ 2xy = 0 \end{cases}$

$$\begin{cases} x=0 \\ -3y=-y^2-3y^2 \end{cases} \quad \vee \quad \begin{cases} y=0 \\ 4x=-x^2-3x^2 \end{cases}$$

$$\begin{cases} x=0 \\ \cancel{-}3y=\cancel{-}4y^2 \end{cases} \qquad \begin{cases} y=0 \\ 4x=-2x^2 \end{cases}$$

$y=0$ $\quad$ $y=3/4$ $\qquad$ $x=0$ $\quad$ $x=-2$

Λ : $\quad (0,0) \quad (0,\tfrac{3}{4}) \quad (0,0) \quad (-2,0)$

$z \Rightarrow (0,0)$	$z_1 = 0$	reale
$\left(0, \frac{3}{4}\right)$	$z_2 = \frac{3}{4} i$	immaginario (puro)
$(-2, 0)$	$z_3 = -2$	reale

2 soluzioni reali, 1 soluzione immaginaria pura

Tema d'esame 9/01/09

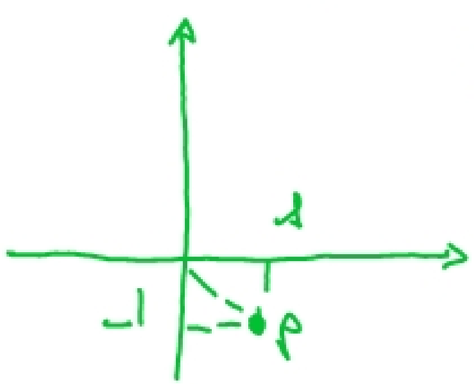
Radici terze del numero complesso $w = 3i\,(1-i)^6$

$w: \quad 3i = 3\,e^{\frac{\pi}{2}i}$

$(1-i)^6 = \left[\sqrt{2}\,e^{\frac{7}{4}\pi i}\right]^6 = \sqrt{2^{\cancel{6}\,3}}\; e^{\frac{7}{\cancel{4}\,2}\cancel{6}^{3}\pi i} =$

$= 2^3\, e^{\frac{21}{2}\pi i} = 2^3\, e^{\frac{\pi}{2} i}$

$1 - i = (1, -1)$



$\rho = \sqrt{2}$

$\theta = \begin{cases} \cos\theta = \frac{1}{\sqrt{2}} = \sqrt{2}/2 \\ \operatorname{sen}\theta = -\frac{1}{\sqrt{2}} = -\sqrt{2}/2 \end{cases}$

$\Downarrow$

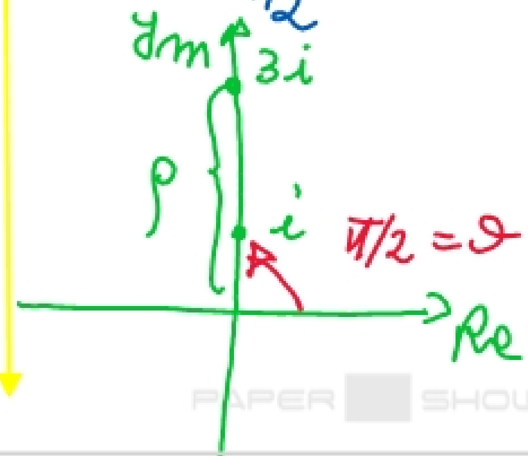
$\theta = 7/4\,\pi$

$3i, \quad Re = 0$
$\quad\;\; Im = 3$

$\rho = 3$

$\theta = \begin{cases} \cos\theta = 0 \\ \operatorname{sen}\theta = 1 \end{cases}$

$\theta = \pi/2$



(si trasforma in notazione esponenziale (prodotto - potenze))

$$w = 3i\,(1-i)^6 = 3\,e^{\pi/2\, i} \cdot 2^3\, e^{\pi/2\, i} = 24\, e^{\pi i}$$

$$\sqrt[3]{w} = \sqrt[3]{24} \cdot e^{\frac{\pi + 2k\pi}{3} i} \qquad k = 0, 1, 2$$

$$\sqrt[3]{3 \cdot 2^3} = 2\sqrt[3]{3}$$

$$w_0 = 2\sqrt[3]{3}\left(\cos\frac{\pi}{3} + i\,\text{sen}\frac{\pi}{3}\right) = \cancel{2}\sqrt[3]{3}\left(\frac{1}{\cancel{2}} + i\frac{\sqrt{3}}{\cancel{2}}\right)$$

$$w_1 = 2\sqrt[3]{3}\left(\cos\pi + i\,\sin\pi\right) = 2\sqrt[3]{3}\,(-1)$$

$$w_2 = 2\sqrt[3]{3}\left(\cos\frac{5}{3}\pi + i\,\text{sen}\frac{5}{3}\pi\right) = \cancel{2}\sqrt[3]{3}\left(\frac{1}{\cancel{2}} - i\frac{\sqrt{3}}{\cancel{2}}\right)$$