Lezione 25: integrali

$$\int \frac{\sinh x}{\cosh x + 1}\, dx =$$

$$= \int \frac{\frac{e^x - e^{-x}}{2}}{\frac{e^x + e^{-x}}{2} + 1}\, dx = \int \frac{e^x - e^{-x}}{e^x + e^{-x} + 2}\, dx =$$

$e^x = t \qquad x = \log t$

$e^{-x} = \frac{1}{t} \qquad dx = \frac{1}{t}\, dt$

$$= \int \frac{t - \frac{1}{t}}{t + \frac{1}{t} + 2} \cdot \frac{1}{t}\, dt = \int \frac{t^2 - 1}{t^2 + 2t + 1}\, dt =$$

$$= \int \frac{(t-1)(t+1)}{(t+1)^2} \cdot \frac{1}{t}\, dt = \int \frac{t-1}{t(t+1)}\, dt =$$

$$\frac{t-1}{t(t+1)} = \frac{A}{t} + \frac{B}{t+1} =$$

$$= \frac{At + A + Bt}{t(t+1)} = \frac{(A+B)t + A}{t(t+1)}$$

$$\begin{cases} A + B = 1 \\ A = -1 \end{cases} \quad \begin{matrix} (-1, 2) \\ (A, B) \end{matrix} \quad \rightarrow \quad = \int \left[-\frac{1}{t} + \frac{2}{t+1} \right] dt =$$

$$= -\log|t| + 2\log|t+1| + c = -\log e^x + 2\log(e^x+1) + c$$

$$\int \frac{x+\sqrt{x-1}}{x-5}\,dx =$$

sostituzione : $\sqrt{x-1} = t$

$$x-1 = t^2$$

$$x = t^2+1$$

$$dx = 2t\,dt$$

$$= \int \frac{\overbrace{t^2+1}^{x} + \overbrace{t}^{\sqrt{x-1}}}{t^2+1-5}\,2t\,dt = 2\int \frac{t^3+t^2+t}{t^2-4}$$

$$\begin{array}{rrrr|l} t^3 & +t^2 & +t & +0 & t^2-4 \\ \hline -t^3 & & +4t & & t+1 \\ \hline // & t^2 & +5t & +0 & \\ & -t^2 & & +4 & \\ \hline & // & 5t & +4 & \end{array}$$

$$= 2\int \left[t+1+\frac{5t+4}{(t-2)(t+2)}\right] dt$$

$$\frac{5t+4}{(t-2)(t+2)} = \frac{A}{t-2} + \frac{B}{t+2} = \frac{A(t+2)+B(t-2)}{(t+2)(t-2)} =$$

$$= \frac{t(A+B)+2A-2B}{(t+2)(t-2)}$$

$$\begin{cases} A+B = 5 \\ \cancel{2}(A-B) = \cancel{4}\ 2 \end{cases} \qquad \begin{cases} 2A = 7 \\ 2B = 3 \end{cases} \qquad \begin{matrix} A = 7/2 \\ B = 3/2 \end{matrix}$$

$$= 2\int (t+1)\,dt + \cancel{2}\int \left[\frac{7}{\cancel{2}} \frac{1}{t-2} + \frac{3}{\cancel{2}} \frac{1}{t+2} \right] dt =$$

$$= t^2 + 2t + 7\log|t-2| + 3\log|t+2| + C =$$

Sostituzione per x

$$= x-1 + 2\sqrt{x-1} + 7\log\left|\sqrt{x-1}-2\right| + 3\log\left|\sqrt{x-1}+2\right| + c$$

$$\int \frac{x}{\sin(x^2)}\,dx = \frac{1}{2}\int \frac{2x}{\sin(x^2)}\,dx =$$

$$= \frac{1}{2}\int \frac{1}{\sin(x^2)}\,d(x^2) =$$

$t = x^2 \qquad dt = 2x\,dx$

$$= \frac{1}{2}\int \frac{1}{\sin t}\,dt =$$

$$= \frac{1}{2}\int \frac{1}{\frac{2\operatorname{tg} y}{1+\operatorname{tg}^2 y}}\,2\,dy = \qquad y = t/2 \qquad dy = \frac{1}{2}\,dt$$

$$= \frac{1}{2}\int \frac{1+\operatorname{tg}^2 y}{\operatorname{tg} y}\,dy = \frac{1}{2}\int \left[\frac{1}{\operatorname{tg} y} + \operatorname{tg} y\right] dy =$$

Param studie:

$$\sin t = \frac{2\operatorname{tg}\frac{t}{2}}{1+\operatorname{tg}^2 t/2}$$

$$\sin t = \frac{2\operatorname{tg} y}{1+\operatorname{tg}^2 y}$$

$y = t/2$

$$\sin x = \frac{2t}{1+t^2}$$

$t = \operatorname{tg} x/2$

$$= \frac{1}{2}\left[\int \frac{\cos y}{\operatorname{sen} y}\,dy + \int -\frac{\operatorname{sen} y}{\cos y}\,dy\right] =$$

$$= \frac{1}{2}\left[\log|\operatorname{sen} y| - \log|\cos y|\right] + c =$$

$$= \frac{1}{2}\log\left|\frac{\operatorname{sen} y}{\cos y}\right| + c_1 =$$

$$= \frac{1}{2}\log|\operatorname{tg} y| + c_1$$

$\left(1 + \operatorname{tg}^2 y = D(\operatorname{tg} y)\right.$ N = derivata del denominatore$)$

$$\int \sqrt{1+x^2}\,dx =$$

$$\boxed{x = \sinh t} \longrightarrow \boxed{\cosh t = \sqrt{1+x^2}}$$

$$dx = \cosh t\,dt$$

$$= \int \overbrace{\cosh t}^{\sqrt{\dots}} \cdot \overbrace{\cosh t\,dt}^{dx} =$$

Relazione fondamentale

$$\boxed{\cosh^2 t - \sinh^2 t = 1}$$

$$\cosh^2 t = 1 + \sinh^2 t$$

$$= \int \cosh^2 t\,dt =$$

$$= \int \left(\frac{e^{t}+e^{-t}}{2}\right)^2 dt =$$

$$= \int \frac{e^{2t}+e^{-2t}+2}{4}\,dt = \frac{1}{4}\int \left(e^{2t}+e^{-2t}+2\right) dt =$$

$$= \frac{1}{4}\left(\frac{1}{2}e^{2t} - \frac{1}{2}e^{-2t} + 2t\right) + C =$$

$$D(e^{2t}) = 2e^{2t}$$

$$D(e^{-2t}) = -2e^{-2t}$$

$$\operatorname{senh} t = \frac{e^{t} - e^{-t}}{2}$$

sostituendo la variabile:

$$x = \operatorname{senh} t$$

$$\cosh t = \sqrt{1+x^2}$$

$$= \frac{1}{4}\left(\frac{e^{2t} - e^{-2t}}{2} + 2t\right) + C$$

$$\cosh^2 t - 1 = \operatorname{senh}^2 t$$

$$= \frac{1}{4}\operatorname{senh}(2t) + \frac{t}{2} + C =$$

$$\operatorname{senh} t = \sqrt{\cosh^2 t - 1}$$

$$e^{2t} - e^{-2t} = (e^{t} - e^{-t})(e^{t} + e^{-t})$$

$$= \frac{1}{2}\operatorname{senh} t \cosh t + \frac{t}{2} + C =$$

funz. inversa del senh t

$$= \frac{1}{2}\, x\sqrt{1+x^2} + \frac{1}{2}\,\underline{\text{sett senh}}\, x + C$$

$$\int \sqrt{x^2-1}\, dx \qquad\qquad x = \cosh t$$

$$dx = \operatorname{senh} t \, dt$$

$$x = \cosh t$$

$$t = \log\left(x + \sqrt{x^2-1}\right) \quad \text{in } [0, +\infty[$$

N.B. gli intervalli di interpretazione per le funzioni inverse!

[N.B. come calcolare la funzione inversa $\quad [0, +\infty[\quad y = \cosh x \nearrow$

(sotto le opportune condizioni dei domini:

$$y = \cosh x = \frac{e^{x} + e^{-x}}{2} \qquad x = f(y)$$

$$2y = e^{x} + \frac{1}{e^{x}} \qquad 2y\, e^{x} = e^{2x} + 1$$

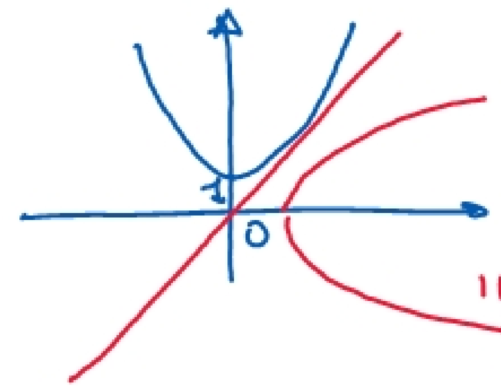
$$e^{2x} - 2y e^{x} + 1 = 0 \rightarrow \left(e^{x}\right)_{1,2} = y \pm \sqrt{y^2 - 1}$$

$$(e^x)_1 = y - \sqrt{y^2-1} \rightarrow x = \log\left(y - \sqrt{y^2-1}\right)$$

$$(e^x)_2 = y + \sqrt{y^2-1} \rightarrow x = \log\left(y + \sqrt{y^2-1}\right)$$

funzione $x = \log\left(y - \sqrt{y^2-1}\right)$ 1° int.

$x = \log\left(y + \sqrt{y^2-1}\right)$ 2° int.



esplicitazione della funzione inversa

$$y = \log\left(x - \sqrt{x^2-1}\right) \quad \text{o} \quad y = \log\left(x + \sqrt{x^2-1}\right)$$

$$\int \sqrt{x^2-1}\, dx = \int \operatorname{senh}^2 t \, dt =$$

$$x = \cosh t \longrightarrow dx = \operatorname{senh} t \, dt$$

$$\cosh^2 t - \operatorname{senh}^2 t = 1$$

$$\cosh^2 t - 1 = \operatorname{senh}^2 t$$

$$\operatorname{senh} t = \sqrt{x^2-1} = \sqrt{\cosh^2 t - 1}$$

vedi esercizio precedente!

$$= \int (\cosh^2 t - 1)\, dt = \int \cosh^2 t \, dt - \int dt =$$

$$= \frac{1}{2} \operatorname{senh} t \cosh t + \frac{t}{2} + c - t =$$

a cui fa seguito la sostituzione - - -

$$\int \frac{2}{(1+\operatorname{tg} x)^2}\,dx =$$

sostituzione per trasformarlo in razionale

$$\operatorname{Tg} x = t$$

$$x = \operatorname{arctg} t$$

$$dx = D(\operatorname{arctg} t) = \frac{1}{1+t^2}\,dt$$

sostituendo

$$= \int \frac{2}{(1+t)^2} \cdot \frac{1}{1+t^2}\,dt$$

FRATTE 2 fattori $(1+t^2)$ non scomp.

$(1+t)^2$ molteplicità soluzione $(t=-1)$

$$\frac{2}{(1+t)^2(1+t^2)} = \underbrace{\frac{A}{1+t} + \frac{B}{(1+t)^2}}_{\substack{\text{molteplicità soluzione} \\ \downarrow \\ \text{"molteplicità frazioni"}}} + \frac{C\cancel{x}t + D}{1+t^2} =$$

$$(1+2t+t^2)$$

$$= \frac{A(1+t)(1+t^2) + B(1+t^2) + (Ct+D)(1+t)^2}{(1+t)^2\,(1+t^2)} =$$

$$= \frac{A(1+t+t^2+t^3) + B(1+t^2) + C(t+2t^2+t^3) + D(1+2t+t^2)}{(1+t)^2\,(1+t^2)} =$$

$$\ldots\; t^3(A+C) + t^2(A+B+2C+D) + t(A+C+2D) + (A+B+D)$$

$$\begin{cases} A+C = 0 \\ A+B+2C+D = 0 \\ A+C+2D = 0 \\ A+B+D = 2 \end{cases} \qquad \begin{cases} A = 1 \\ B = 1 \\ C = -1 \\ D = 0 \end{cases}$$

$$= \int \frac{1}{1+t}\,dt + \int \frac{1}{(1+t)^2}\,dt - \frac{1}{2}\int \frac{2t}{1+t^2}\,dt =$$

$$= \log|1+t| - \frac{1}{1+t} - \frac{1}{2}\log(1+t^2) + c =$$

$t = \operatorname{tg} x$

sostituendo

$$= \log|\operatorname{tg} x + 1| - \frac{1}{1+\operatorname{tg} x} - \frac{1}{2}\log(1+\operatorname{tg}^2 x) + c$$

$$\int \frac{\cos x - 3}{\sin^2 x - \cos^3 x + 1} \sin x \, dx$$

sostituzione: $\cos x = t$

$$\int \frac{1}{4\operatorname{sen} x + 3\cos x}\, dx$$

sostituzione: $\operatorname{tg}\frac{x}{2} = t$

parametriche

$$\operatorname{sen} x = \frac{2\operatorname{tg}\frac{x}{2}}{1+\operatorname{tg}^2\frac{x}{2}} = \frac{2t}{1+t^2}$$

$$\cos x = \frac{1-\operatorname{tg}^2\frac{x}{2}}{1+\operatorname{tg}^2\frac{x}{2}} = \frac{1-t^2}{1+t^2}$$