Lezione 9: successioni

1- fattoriale

$$n! = n\cdot(n-1)\cdot(n-2)\cdots\underbrace{(n-n+1)}_{1} \quad n\in N$$

indice il prodotto di un numero naturale n per tutti i suoi precedenti (numeri)

n: $1! = 1$

$2! = 2\times 1 = 2$ $\qquad n>1 \quad n!$ è pari

$3! = 3\times 2\times 1 = 6$

$4! = 4\times 3\times 2\times 1 = 24$

...

$[\,0! = 1\,]$

$$a_n = \frac{(n+3)!}{n!} = \frac{(n+3)(n+2)(n+1)\cancel{n!}}{\cancel{n!}} = (n+3)(n+2)(n+1) = \dots$$

polinomio 3° grado

$$\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} \frac{(n+3)!}{n!} = \frac{\infty}{\infty} \qquad \text{F. I.}$$

$$= \lim_{n\to+\infty} \frac{(n+3)(n+2)(n+1)\cancel{n!}}{\cancel{n!}} =$$

$$= \lim_{n\to+\infty} n\Big(1+\underbrace{\tfrac{3}{n}}_{0}\Big)\, n\Big(1+\underbrace{\tfrac{2}{n}}_{0}\Big)\, n\Big(1+\underbrace{\tfrac{1}{n}}_{0}\Big) \simeq$$

$$\simeq \lim_{n\to+\infty} n^3 = +\infty$$

COMPORTAMENTO ASINTOTICO

$$a_n = \log(n!) - \log[(n+1)!]$$

$$\lim_{n \to +\infty} a_n = \lim_{n \to +\infty} \Big\{ \underbrace{\log(n!)}_{+\infty} - \underbrace{\log[(n+1)!]}_{+\infty} \Big\} =$$

$+\infty - \infty$ F. I.

$$= \lim_{n \to +\infty} \log \frac{n!}{(n+1)!} = \lim_{n \to \infty} \log \underbrace{\frac{\cancel{n!}}{(n+1)\,\cancel{n!}}}_{\rightsquigarrow 0} = -\infty$$

coefficienti binomiali:

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} \qquad 0 \leq k \leq n \qquad k, n \in \mathbb{N}$$

$$\binom{n}{k} = \binom{n}{n-k}$$

$$\binom{n}{1} = n \qquad \forall n \geq 1 \qquad \left[= \frac{n!}{1!\,(n-1)!} = n \right]$$

$$\binom{n}{0} = \binom{n}{n} = 1$$

$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1} \qquad \binom{n}{k+1} = \frac{n-k}{k+1}\binom{n}{k}$$

N. B.

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k$$

$$(1+x)^0 \qquad 1$$

$$(1+x)^1 \qquad 1 + x$$

$$1 + 1$$

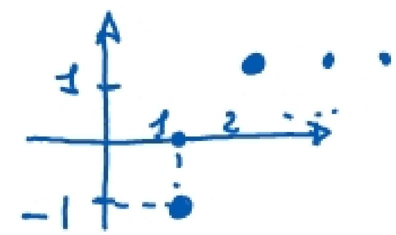
$$(1+x)^2 \qquad 1 + 2x + x^2$$

$$1 + 2 \qquad 2 + 1$$

$$(1+x)^3 \qquad 1 + 3x + 3x^2 + x^3$$

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

$$a_n = (-1)^{n!} = \begin{cases} -1 & n = 1 \quad n! = 1! = 1 \\ 1 & n \geq 2 \quad n! \text{ pari} \end{cases}$$



$$a_n = n^{(-1)^n} = \begin{cases} n & n = 2m \quad (\text{pari}) \\ \dfrac{1}{n} & n = 2m+1 \quad (\text{dispari}) \end{cases}$$

divergente $+\infty$

convergente $\to 0$

$$b_n = (-1)^n = \begin{cases} 1 & n = 2m \quad \text{pari} \\ -1 & n = 2m+1 \quad \text{dispari} \end{cases}$$

a_n irregolare, a termini positivi

CONFRONTO / ORDINE DI INFINITO

$$a_n = \log_a n \qquad a > 1$$

$$b_n = n^{\alpha} \qquad \alpha \in \mathbb{N}$$

$$c_n = a^n \qquad a > 1$$

$$d_n = n!$$

$$e_n = n^n$$

per $n \to +\infty$: $\log_a n < n^{\alpha} < a^n < n! < n^n$

$$\lim_{n \to +\infty} \frac{\log_a n}{n!} = 0 \qquad \lim_{n \to +\infty} \frac{n!}{\log_a n} = +\infty$$

CONFRONTO TRA POLINOMI

$$a_n = \frac{n^2+5}{n+1} \qquad \text{grado } N > \text{grado } D.$$

$$\lim_{n\to+\infty} \frac{n^2+5}{n+1} = \frac{\infty}{\infty} \quad \text{F.I.}$$

$$= \lim_{n\to+\infty} \frac{n^2\left(1+\frac{5}{n^2}\right)}{n\left(1+\frac{1}{n}\right)} = \lim_{n\to+\infty} n = +\infty$$

$$b_n = \left(\frac{2-3n}{2n+1}\right)^3 \qquad \text{grado } N = \text{grado } D$$

$$\lim_{n\to+\infty} \left(\frac{2-3n}{2n+1}\right)^3 = \text{F.I.} \quad \frac{\infty}{\infty}$$

$$= \lim_{n\to+\infty} \left(\frac{\cancel{n}\,(\underbrace{\tfrac{2}{n}}_{\to 0} - 3)}{\cancel{n}\,(2 + \underbrace{1/n}_{\to 0})}\right)^3 = \left(-\frac{3}{2}\right)^3$$

$\Rightarrow$ confronto fra i coefficienti di grado massimo.

$$c_n = \frac{n+3}{5n^2-1} \qquad \text{grado N} < \text{grado D}$$

$$\lim_{n \to +\infty} \frac{n+3}{5n^2-1} = \text{F.I. } \frac{\infty}{\infty}$$

$$= \lim_{n \to +\infty} \frac{\cancel{n}\left(1+\frac{3}{n}\right)}{n^{\cancel{2}}\left(5-\frac{1}{n}\right)} \sim \lim_{n \to +\infty} \frac{1}{n} = 0$$

$\Rightarrow$ se prevale il grado del denominatore, vale 0

$$d_n = \underbrace{\frac{n^2+5}{n+1}}_{+\infty} + \underbrace{\operatorname{sen}(\underbrace{\log n}_{+\infty})}_{[-1,\,1]} \qquad \lim_{n \to +\infty} d_n = +\infty$$

$$e_n = \operatorname{sen}(\log n) \qquad \lim_{n\to+\infty} \operatorname{sen}(\log n) = \nexists$$

$$-1 \le \operatorname{sen}(\log n) \le 1$$

$$f_n = \underbrace{\frac{n+3}{5n^2-1}}_{\to 0} + \underbrace{\operatorname{sen}(\log n)}_{[-1,1]} \qquad \lim_{n\to+\infty} f_n = \nexists$$

$$g_n = \underbrace{\frac{2-3n}{2n+1}}_{\to -3/2} + \underbrace{\operatorname{sen}(\log n)}_{[-1,1]} \qquad \lim_{n\to+\infty} g_n = \nexists$$

- $\lim_{n \to +\infty} \sqrt[n]{n} = \lim_{n \to +\infty} n^{\frac{1}{n}} = \infty^0 \; ?$

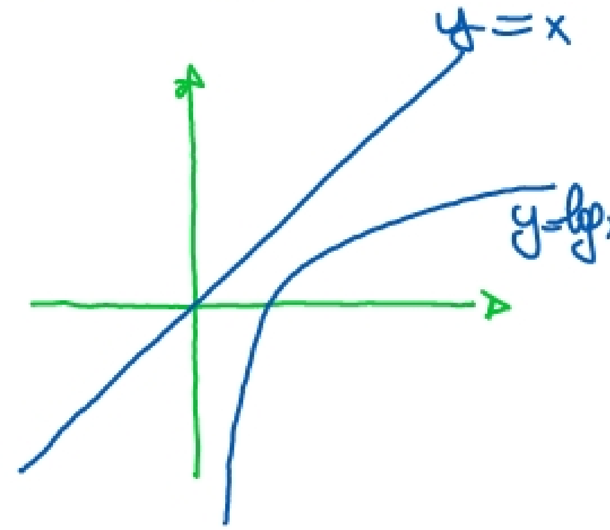
$$= \lim_{n \to +\infty} e^{\log n^{\frac{1}{n}}} =$$

← Passaggio all'esponenziale: $e^{\log(\;)}$

$$= \lim_{n \to +\infty} e^{\frac{1}{n} \cdot \log n} =$$

$$= \lim_{n \to +\infty} e^{\frac{\log n}{n}} = \qquad \frac{\log n}{n} \to 0$$

$$= e^0 = 1$$



• $\lim_{n \to +\infty} \frac{n!}{n^n} = \frac{\infty}{\infty} \Rightarrow 0$ per confronto

$$\frac{n!}{n^n} = \frac{n\,(n-1)\cdots\cdots 3\cdot 2\cdot \overset{(n-n+1)}{1}}{\underbrace{n\cdot n\cdot n\cdot n\cdots}_{n \text{ fattori}}} \le \frac{1}{n}$$

$\frac{n}{n} = 1$; $\frac{n-1}{n} < 1$; .... ; $\frac{3}{n} < 1$; $\frac{2}{n} < 1$; $\frac{1}{n} < 1$

$\forall\, n > 2$

$$\Rightarrow \quad \underset{\substack{\downarrow \\ 0}}{0} < \underset{\substack{\downarrow \\ 0}}{\frac{n!}{n^n}} \le \underset{\substack{\downarrow \\ 0}}{\frac{1}{n}}$$

$\lim_{n \to +\infty} \frac{1}{n} = 0$

TEO. DEI CARABINIERI

$$\bullet \lim_{n \to +\infty} \frac{n + \overset{[-1,1]}{\sin n}}{\underbrace{\underset{[0,\pi/2]}{\operatorname{arctg} n}}_{\pi/2} - n} = \frac{\infty}{\infty}$$

$$= \lim_{n \to +\infty} \frac{\cancel{n}\left(1 + \overbrace{\frac{\sin n}{n}}^{\to 0}\right)}{\cancel{n}\left(\underbrace{\frac{\operatorname{arctg} n}{n}}_{\to 0} - 1\right)} =$$

$$= -1$$

$$\lim_{n \to +\infty} \frac{n \sin n}{\underbrace{n^2+1}_{>0}} \qquad \sin n \to [-1, 1]$$

$$-1 \leq \sin n \leq 1 \qquad \times (n) > 0$$

$$-n \leq n \sin n \leq 1 \qquad : (n^2+1) > 0$$

$$-\frac{n}{n^2+1} \leq \frac{n \sin n}{n^2+1} \leq \frac{n}{n^2+1} \qquad \forall n \in \mathbb{N}$$

$$\lim_{n \to +\infty} \qquad 0 \qquad\qquad 0 \qquad\qquad 0$$

Teo. dei Carabinieri

$$\bullet \lim_{n\to+\infty} \frac{\log[(n+3)!] - \log(n!)}{\log(2n^6)} =$$

$$= \lim_{n\to+\infty} \frac{\log \dfrac{(n+3)!}{n!}}{\log 2 + \log n^{6}} =$$

$$= \lim_{n\to+\infty} \frac{\log \left[(n+3)(n+2)(n+1) \dfrac{\cancel{n!}}{\cancel{n!}} \right]}{\log 2 + 6 \log n} =$$

$$= \lim_{n\to+\infty} \frac{\log \left[n^3 \underbrace{\left(1+\frac{3}{n}\right)^{\to 0} \left(1+\frac{2}{n}\right)^{\to 0} \left(1+\frac{1}{n}\right)^{\to 0}}_{\text{TRASCURABILI}} \right]}{\log n \left(6 + \underbrace{\dfrac{\log 2}{\log n}}_{\to 0} \right)} = \lim_{n\to+\infty} \frac{3 \log n}{6 \log n} = \frac{1}{2}$$

$$\bullet \lim_{n\to+\infty} \frac{(n-1)!\; n^{n+1} - (n+1)!\; n^{n-1}}{n^n\left((n-1)! + \log n\right)} =$$

$$(n+1)! = (n+1)\,n\,\underline{(n-1)!}$$

$$n^{n+1} = n^n \cdot n = n^{n-1} \cdot n \cdot n = \underline{n^{n-1}}\, n^2$$

$$= \lim_{n\to+\infty} \frac{\cancel{(n-1)!}\; \cancel{n^{n-1}} \left[n^2 - n(n+1)\right]}{n^{\cancel{n}}\; \cancel{(n-1)!} \left(1 + \underbrace{\frac{\log n}{(n-1)!}}_{\to 0}\right)} =$$

$$= \lim_{n\to+\infty} \frac{\cancel{n^2} - \cancel{n^2} - n}{n} = -\underline{1}$$