SERIE 3

CONVERGENZA ASSOLUTA $\Rightarrow$ CONVERGENZA SEMPLICE

def. • $\sum a_n$ si dice assolutamente convergente se $\sum |a_n|$ converge.

• $\sum a_n$ si dice semplicemente convergente se $\sum |a_n|$ non converge.

Teo. Se $\sum |a_n|$ converge $\Rightarrow$ $\sum a_n$ converge

SERIE A SEGNO ALTERNO

$$\sum_{n=0}^{+\infty} (-1)^n a_n$$

CRITERIO DI LEIBNIZ

Sia $\{a_n\}$ reale, non crescente e infinitesima

$\Rightarrow$ la serie $\displaystyle\sum_{n=0}^{+\infty} (-1)^n a_n$ converge (semplicemente)

3 ipotesi:

- $\forall n \in \mathbb{N} \quad a_n \in \mathbb{R}$
- $a_n \geq a_{n+1} \qquad \forall n \in \mathbb{N}$
- $\displaystyle\lim_{n \to +\infty} a_n = 0$

es 1 $\quad \sum_{n=1}^{+\infty} \frac{(n+1)\,\boxed{\sin n}}{n^{7/3}+\log n}$ serie a segno non definito
($-1 \leq \sin n \leq 1$)

$[0,1]$

$$|a_n| = \left| \frac{(n+1)\cdot \sin n}{n^{7/3}+\log n} \right| = \frac{(n+1)}{n^{7/3}+\log n} \, |\sin n| \leq$$

$$\leq \frac{n+1}{n^{7/3}+\log n}$$ (successione maggiorante $\forall n$)

$|a_n| \leq b_n$

comportamento asintotico

$$b_n = \frac{n+1}{n^{7/3}+\log n} \simeq \frac{n}{n^{7/3}} = \frac{1}{n^{7/3-1}} = \frac{1}{n^{4/3}} \qquad \alpha = 4/3 > 1$$

conv.

$\sum \frac{1}{n^{4/3}}$ conv $\overset{\text{ASINT.}}{\Longrightarrow}$ $\sum b_n$ conv. $\overset{\text{magg.}}{\Longrightarrow}$ $\sum |a_n|$ conv. $\underset{\text{def.}}{\Longrightarrow}$

$\Longrightarrow \sum a_n$ conv. ass. $\underset{\text{Teorema}}{\Longrightarrow}$ conv. semp.

es 2 $\displaystyle\sum_{n=1}^{+\infty} \left(e^{\frac{1}{\sqrt{n}}} - 1\right) (-1)^n$ serie a segni alterni

$$\left|(-1)^n \left(e^{\frac{1}{\sqrt{n}}} - 1\right)\right| = e^{\frac{1}{\sqrt{n}}} - 1$$

criterio del confronto asintotico

$$\lim_{n\to+\infty} \frac{e^{\frac{1}{\sqrt{n}}} - 1}{\frac{1}{\sqrt{n}}} = 1$$

$$\sum \frac{1}{\sqrt{n}} = \sum \frac{1}{n^{1/2}}$$ divergente $\alpha = \frac{1}{2} < 1$

$$\left(e^{\frac{1}{\sqrt{n}}} - 1\right) \simeq \frac{1}{\sqrt{n}} \quad \Rightarrow \quad \sum |a_n| \text{ diverge}$$

convergenza semplice?

se si utilizza il criterio di Leibnitz:

① $\lim_{n\to+\infty}\left(e^{\frac{1}{\sqrt{n}}}-1\right)=0 \Rightarrow a_n$ è infinitesima

② non crescenza:

$$n < n+1$$
$$\sqrt{n} < \sqrt{n+1}$$
$$\frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}}$$
$$e^{\frac{1}{\sqrt{n}}} > e^{\frac{1}{\sqrt{n+1}}}$$
$$e^{\frac{1}{\sqrt{n}}}-1 > e^{\frac{1}{\sqrt{n+1}}}-1 \Rightarrow a_n > a_{n+1} \quad \forall n\in\mathbb{N}$$

③ è a valori reali

$\Rightarrow$ la $\sum_{n=1}^{+\infty}\left(e^{\frac{1}{\sqrt{n}}}-1\right)$ converge.

$$\sum_{n=2}^{+\infty} (-1)^n \frac{2^n}{n!}$$ converge assolutamente?

S. segni alterni

$$b_n = \left| (-1)^n \frac{2^n}{n!} \right| = \frac{2^n}{n!}$$

$$\lim_{n\to+\infty} \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} = \lim_{n\to+\infty} \frac{\cancel{2^n} \cdot 2 \cdot \cancel{n!}}{(n+1)\cancel{n!} \cdot \cancel{2^n}} = 0$$

conv. assoluta $\Rightarrow$ convergenza semplice

$$\sum_{n=1}^{+\infty} \frac{3^n \, n!}{n^n}$$ converge?

serie termini positivi

$$\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to+\infty} \frac{3^{n+1}\cdot(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{3^n \, n!} =$$

$$= \lim_{n\to+\infty} \frac{\cancel{3^n}\cdot 3 \cdot \cancel{(n+1)}\,\cancel{n!}\cdot n^n}{(n+1)^n \cdot \cancel{(n+1)}\,\cancel{3^n}\,\cancel{n!}} =$$

$$= 3 \lim_{n\to+\infty} \left(\frac{n}{n+1}\right)^n = \frac{3}{e} > 1$$ diverge

$$\sum_{n=2}^{+\infty} (-1)^n \frac{1}{\log(7^n+2)}$$

converge assolutamente?

$$\lim_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to+\infty} \left| \frac{\overset{\cancel{(-1)^n}\cdot(-1)}{(-1)^{n+1}}}{\log(7^{n+1}+2)} \cdot \frac{\log(7^n+2)}{\underbrace{\cancel{(-1)^n}}} \right| =$$

$$= \lim_{n\to+\infty} \frac{\log(7^n+2)}{\log(7^{n+1}+2)} =$$

$$= \lim_{n\to+\infty} \frac{\log\left[7^n\left(1+\overset{\to 0}{\frac{2}{7^n}}\right)\right]}{\log\left[7^{n+1}\left(1+\underset{\to 0}{\frac{2}{7^{n+1}}}\right)\right]} \simeq \lim_{n\to+\infty} \frac{\log 7^n}{\log 7^{n+1}} =$$

$$= \lim_{n\to+\infty} \frac{n \log 7}{(n+1)\log 7} = 1$$

non converge assolutamente.

Si utilizza il criterio di Leibnitz: $a_n = \frac{1}{\log(7^n+2)}$

1. $\lim_{n \to +\infty} \frac{1}{\log(7^n+2)} = 0$ infinitesima

2. $y = \frac{1}{\log(7^x+2)}$ la funzione è decrescente all'aumentare di x, poiché aumenta il denominatore

$$n < n+1$$
$$7^n < 7^{n+1}$$
$$7^n+2 < 7^{n+1}+2$$
$$\frac{1}{7^n+2} > \frac{1}{7^{n+1}+2} \Rightarrow \frac{1}{\log(7^n+2)} > \frac{1}{\log(7^{n+1}+2)}$$
$$a_n > a_{n+1}$$

3. a valori reali

$\Rightarrow$ per il criterio converge semplicemente

• Calcolare il valore della seguente serie:

$$\sum_{n=1}^{+\infty} \frac{2n+1}{(2n)!} =$$

$$= \sum_{n=1}^{+\infty} \frac{2n \cdot 1^{2n}}{(2n)!} + \sum_{n=1}^{+\infty} \frac{1}{(2n)!} =$$

N.B. $\displaystyle\sum_{n=0}^{+\infty} \frac{x^{2n}}{(2n)!} = \cosh x$ $\quad x=1$

$$a_0 = \cosh 0 = \frac{e^0+e^{-0}}{2} = \frac{1+1}{2} = 1$$

$\displaystyle\sum_{n=0}^{+\infty} \frac{x^{2n+1}}{(2n+1)!} = \sinh x$ $\quad x=1$

$$b_0 = \sinh 0 = \frac{e^0-e^{-0}}{2} = \frac{1-1}{2} = 0$$

$$\sum_{n=1}^{+\infty} \frac{2n}{(2n)!} = \sum_{n=1}^{+\infty} \frac{\cancel{2n}}{(\cancel{2n})(2n-1)!} = \sum_{n=1}^{+\infty} \frac{1}{(2n-1)!}$$

$m \in \mathbb{N}$:

$$2n-1 = 2m+1$$

$$2m+2 = 2n$$

$$n = m+1 \qquad n=1 \qquad m+1=1$$

$$m=0$$

$$\sum_{n=1}^{+\infty} \frac{1}{(2n-1)!} = \sum_{m=0}^{+\infty} \frac{1}{(2(m+1)-1)!} = \sum_{m=0}^{+\infty} \frac{1}{(2m+2-1)!} =$$

$$= \sum_{m=0}^{+\infty} \frac{1}{(2m+1)!} 1^{2m+1} = \sinh 1 \qquad x=1$$

$$= \sinh 1 + \cosh 1 - \underbrace{1}_{a_0} = \frac{e^1 - e^{-1}}{2} + \frac{e^1 + e^{-1}}{2} - 1 =$$

$$= \frac{e^1 - \cancel{e^{-1}} + e^1 + \cancel{e^{-1}} - 2}{2} = e - 1$$

• Calcolare il valore delle seguente serie:

$$\sum_{n=2}^{+\infty} (-1)^{n+1} \frac{4^n}{(2n-1)!} =$$

$$= \sum_{n=2}^{+\infty} \frac{(-1)^{n+1}\, 2^{2n}}{(2n-1)!} =$$

$$2n-1 = 2m+1$$

$$2m+2 = 2n$$

$$n = m+1$$

$$n=2 \qquad m=1$$

$$= \sum_{m=1}^{+\infty} \frac{(-1)^{m+2}\, 2^{2(m+1)}}{(2m+2-1)!} =$$

$$1 \downarrow 0$$

$$= \sum_{m=0}^{+\infty} \frac{(-1)^{m+2}\, 2^{2m+2}}{(2m+1)!} - a_0 = \sum_{m=0}^{+\infty} \frac{(-1)^m \cdot \overset{+1}{(-1)^2}\, 2^{2m+1} \cdot 2}{(2m+1)!} - 4 =$$

$$= 2\underbrace{\sum_{m=0}^{+\infty} \frac{(-1)^m\, 2^{2m+1}}{(2m+1)!}} - 4 = 2\sin 2 - 4 = 2(\sin 2 - 2)$$

$$\sin x = \sum_{n=0}^{+\infty} \frac{(-1)^n\, x^{2n+1}}{(2n+1)!} \quad \text{con } x=2$$

- Calcolare il valore della seguente serie

$$\sum_{n=1}^{+\infty} (-1)^n \frac{2^{n+1}}{3^{n+2}\, n!} =$$

$$= \sum_{n=1}^{+\infty} (-1)^n \frac{2^{n+1}}{3^{n+1}\cdot 3\cdot n!} = \sum_{n=1}^{+\infty} (-1)^n \left(\frac{2}{3}\right)^{n+1}\cdot \frac{1}{3\cdot n!} =$$

$$= \frac{1}{3} \sum_{n=1}^{+\infty} \frac{(-1)^n \left(\frac{2}{3}\right)^{n+1}}{n!} = \frac{1}{3} \sum_{n=1}^{+\infty} \frac{(-1)^n \left(\frac{2}{3}\right)^{n}}{n!}\cdot\frac{2}{3} =$$

$$= \frac{2}{9}\left(\sum_{n=0}^{+\infty} \frac{\left(-\frac{2}{3}\right)^n}{n!} - \underbrace{\frac{(-1)^0\left(\frac{2}{3}\right)^0}{0!}}_{a_0}\right) =$$

$$= \frac{2}{9}\left(e^{-2/3} - 1\right)$$

$$\sum_{n=0}^{+\infty} \frac{x^n}{n!} = e^x$$

con $x = -2/3$

- Data la serie definita dalla successione :

$$\begin{cases} a_0 = \alpha > 0 \\ a_{n+1} = \dfrac{a_n}{2+a_n} & n \geq 0 \end{cases}$$

determinarne il carattere

- $a_n > 0 \quad \forall n \in \mathbb{N}$

- monotonia $\quad a_{n+1} - a_n = \dfrac{a_n}{2+a_n} - a_n = a_n\left(\dfrac{1}{2+a_n} - 1\right) =$

$$= \frac{-a_n(a_n+1)}{2+a_n} < 0 \Rightarrow a_{n+1} < a_n$$

$\Rightarrow a_n$ è decrescente e inferiormente limitata $(\inf = 0)$

- calcolo del limite

$$L = \frac{L}{2+L} \qquad \lim_{n\to+\infty} a_{n+1} = L = \lim_{n\to+\infty} a_n$$

$$2L + L^2 = L \qquad L^2 + L = 0 \qquad L(L+1) = 0 \qquad \begin{array}{l} L_1 = 0 \text{ acc.} \\ L_2 = -1 \text{ non acc} \end{array}$$

$\Rightarrow \lim_{n\to+\infty} a_n = 0$ (infinitesimo)

$\Rightarrow$ verifichiamo la convergenza con il criterio del rapporto asintotico:

$$\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to+\infty} \frac{\frac{a_n}{2+a_n}}{a_n} =$$

$$= \lim_{n\to+\infty} \frac{1}{2+a_n} = \frac{1}{2+\lim_{n\to+\infty} a_n} = \frac{1}{2+0} = \frac{1}{2} < 1$$

$\Rightarrow$ converge