FORMULE DI INTEGRAZIONE

REGOLE DI INTEGRAZIONE

$$\int (A f(x) + B g(x)) \, dx = A \int f(x) \, dx + B \int g(x) \, dx$$

$$\int f'(g(x)) \cdot g'(x) \, dx = f(g(x)) + c$$

$$\int u(x) \cdot dv(x) = u(x) \cdot v(x) - \int v(x) \, du(x)$$ (per parti)

$$\int_a^b f'(x) \, dx = f(b) - f(a)$$

$$\frac{d}{dx} \int_a^x f(t) \, dt = f(x)$$

INTEGRALI ELEMENTARI

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + c \qquad n \neq -1$$

$$\int \frac{dx}{x} = \int \frac{1}{x} \, dx = \ln|x| + c$$

$$\int e^x \, dx = e^x + c$$

$$\int a^x \, dx = \frac{a^x}{\ln a} + c \qquad (a>0,\ a\neq 1)$$

$$\int \operatorname{sen} x \, dx = -\cos x + c$$

$$\int \cos x \, dx = \operatorname{sen} x + c$$

$$\int \sec^2 x \, dx = \tan x + c \qquad \left[\sec x = \frac{1}{\cos x}\right]$$

$$\int \operatorname{cosec}^2 x \, dx = -\operatorname{cotg} x + c \qquad \left[\operatorname{cosec} x = \frac{1}{\operatorname{sen} x}\right]$$

$$\int \sec x \cdot \operatorname{Tg} x \, dx = \sec x + c$$

$$\int \operatorname{cosec} x \cdot \operatorname{ctg} x \, dx = -\operatorname{cosec} x + c$$

$$\int \operatorname{tg} x \, dx = \ln|\sec x| + c$$

$$\int \operatorname{ctg} x \, dx = \ln|\operatorname{sen} x| + c$$

$$\int \sec x \, dx = \ln|\sec x + \operatorname{tg} x| + c$$

$$\int \operatorname{cosec} x \, dx = \ln|\operatorname{cosec} x - \operatorname{ctg} x| + c$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \operatorname{arcsen} \frac{x}{|a|} + c$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \operatorname{arctg} \frac{x}{a} + c$$

$$\int \frac{dx}{a^2-x^2} = \frac{1}{2a} \ln \left| \frac{x+a}{x-a} \right| + C$$

$$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a} \operatorname{arcsec} \frac{|x|}{a} + C$$

$$\int x \sin x \, dx = \sin x - x \cos x + C$$

$$\int x \cos x \, dx = \cos x + x \sin x + C$$

$$\int \sinh x \, dx = \cosh x + C$$

$$\int \cosh x \, dx = \sinh x + C$$

$$\int \operatorname{tgh} x \, dx = \ln (\cosh x) + C$$

$$\int \operatorname{ctgh} x \, dx = \ln | \sinh x | + C$$

INTEGRAZIONI ELEMENTARI

① $\int 5x^2\,dx = 5\cdot\frac{1}{2+1}x^{2+1}+c = \frac{5}{3}x^3+c$

$\int \sqrt[4]{x^3}\,dx = \int x^{\frac{3}{4}}dx = \frac{1}{7/4}x^{7/4}+c = \frac{4}{7}x^{7/4}+c$

$\int \frac{x}{\sqrt[7]{x^3}}\,dx = \int x^{1-\frac{3}{7}}dx = \int x^{4/7}dx = \frac{7}{11}x^{\frac{11}{7}}+c$

$\int \left(\frac{9}{x^7} + 8x^2\sqrt[7]{x^5}\right)dx = \int 9x^{-7}dx + \int 8x^{\frac{19}{7}}dx =$

$= \frac{9}{-6}x^{-6} + c_1 + 8\cdot\frac{7}{26}x^{\frac{26}{7}} + c_2 \quad +c \ldots$

② $\int \frac{1}{x-1}\,dx = \ln|x-1|+c$

$D(5+2x)$

$\int \frac{2}{2}\,\frac{1}{5+2x}\,dx = \frac{1}{2}\ln|5+2x|+c$

$D(x^3+5x+1) = 3x^2+5$ RN

$\int \frac{3x^2+5}{x^3+5x+1}\,dx = \ln|x^3+5x+1|+c$

③ $\int (e^x+5)^3\cdot \underbrace{e^x\,dx}_{\substack{d(e^x)\\ d(e^x+5)}} = \frac{1}{4}(e^x+5)^4+c$

$\int \frac{5}{5}e^{5x-2}\,dx = \frac{1}{5}e^{5x-2}+c$

$D(e^{5x-2}) = 5e^{5x-2}$

$\int \frac{2e^{\sqrt{x}}}{2\sqrt{x}}\,dx = 2e^{\sqrt{x}}+c$

$D(\sqrt{x}) = \frac{1}{2\sqrt{x}}$

$\int \frac{e^x}{e^x+1}\,dx = \lg(e^x+1)+c$

$D(e^x+1) = e^x \qquad e^x+1>0$

$\int \frac{2}{2}\,\frac{1}{x\sqrt{\ln x}}\,dx = 2\sqrt{\lg x}+c$

$D(\sqrt{\ln x}) = \frac{1}{2\sqrt{\ln x}}\cdot\frac{1}{x}$

① $\boxed{\int x^n dx = \frac{1}{n+1}x^{n+1}+c}$ ② $\boxed{\int \frac{1}{x}dx = \lg|x|+c}$ ③ $\boxed{\int e^x dx = e^x}$

$$\int \frac{2 \cdot 1}{2\, x \ln x}\, dx = 2 \ln\left(\sqrt{\ln x}\right) + C$$

$$D\left[\ln\left(\sqrt{\ln x}\right)\right] = \underbrace{\frac{1}{\sqrt{\ln x}} \cdot \frac{1}{2\sqrt{\ln x}}} \cdot \frac{1}{x}$$

$$\int \frac{3\, e^{3x}}{3\,(2 + e^{3x})}\, dx = \frac{1}{3} \ln\left(2 + e^{3x}\right) + C$$

$$D(2 + e^{3x}) = 3 e^{3x}$$

$$\int \frac{2 \operatorname{sen} x \cdot \cos x}{2}\, dx = \frac{2}{2} \int \operatorname{sen} 2x\, dx = -\frac{1}{4} \cos 2x + C_1$$

$$= \frac{1}{2} \int 2 \underline{\operatorname{sen} x} \cos x\, dx = \frac{1}{2} \operatorname{sen}^2 x + C$$

$$D(\operatorname{sen} x) = \cos x$$

$$\int tg^5 x \cdot \underbrace{\frac{1}{\cos^2 x}}_{D(tg x) = \frac{1}{\cos^2 x}}\, dx = \frac{1}{6}\, Tg^6 x + C$$

$$\int \underline{2x} \operatorname{sen} x^2\, \underline{dx} = -\cos x^2 + C$$

$$D(x^2) = 2x$$

$$\int \frac{3x^2}{3} \cos x^3\, \underline{dx} = \frac{1}{3} \operatorname{sen} x^3 + C$$

$$D(x^3) = 3x^2$$

$$-\int -\cos^5 x \cdot \overline{\operatorname{sen} x\, dx} = -\frac{1}{6} \cos^6 x + C$$

$$D(\cos x) = -\operatorname{sen} x$$

$$\int \underbrace{\frac{1}{\cos^2 x^3}}_{D(tg\, t)} \frac{\overbrace{3x^2\, dx}^{D(x^3)}}{3} = \frac{1}{3}\, Tg\, x^3 + C$$

$$\int \frac{tg^3 x}{\cos^2 x}\, dx = \frac{1}{4}\, Tg^4 x + C$$

$$D(tg x)$$

$$-\frac{1}{7} \int -\frac{1}{\operatorname{sen}^2 7x}\, 7\, dx = -\frac{1}{7}\, ctg\, 7x + C$$

$$D(ctg x) = -\frac{1}{\operatorname{sen}^2 x}$$

$$\int \underline{\cos x} \sqrt{1 + \operatorname{sen} x}\; \underline{dx} = \frac{2}{3} (1 + \operatorname{sen} x)^{3/2} + C$$

$$\sqrt{1 + \operatorname{sen} x} = (1 + \operatorname{sen} x)^{1/2}$$

$$D(1 + \operatorname{sen} x) = \cos x$$

$$\int \frac{1}{1+\ln^2 x} \cdot \frac{1}{x}\, dx = \operatorname{arctg}(\ln x) + C$$

$\frac{1}{x} = D(\ln x)$

$$\int 2\frac{\cos\sqrt{x}}{2\sqrt{x}}\, dx = 2 \operatorname{sen}\sqrt{x} + C$$

$$-\int (1+\cos x)^5 (-\operatorname{sen} x\, dx) = -\frac{1}{6}(1+\cos x)^6 + C$$

$D(1+\cos x) = -\operatorname{sen} x$

$$\int e^{\operatorname{sen} x} \underbrace{\cos x\, dx}_{D(\operatorname{sen} x)} = e^{\operatorname{sen} x} + C$$

$$\int \frac{\cos(\ln x)}{x}\, dx = \operatorname{sen}(\ln x) + C$$

$\frac{1}{x}\,dx$: $D(\ln x)$

$$\int \frac{\operatorname{arctg}^7 x}{1+x^2}\, dx = \frac{1}{8} \operatorname{arctg}^8 x + C$$

$\frac{1}{1+x^2}\,dx$: $D(\operatorname{arctg} x)$

$$\int \frac{1}{1+\operatorname{sen}^2 x} \cdot \underbrace{\cos x\, dx}_{D(\operatorname{sen} x)} = \operatorname{arctg}(\operatorname{sen} x) + C$$

$$\int \frac{1-\operatorname{sen} x}{x+\cos x}\, dx = \ln|x+\cos x| + C$$

$D(\operatorname{arctg} x) = \frac{1}{1+x^2}$

$D(x+\cos x) = 1 - \operatorname{sen} x$

$$\int \frac{\operatorname{arcsen}^3 x}{\sqrt{1-x^2}}\,dx = \frac{1}{4}\operatorname{arcsen}^4 x + c$$

$$D(\operatorname{arcsen} x) = \frac{1}{\sqrt{1-x^2}}$$

$$\int \frac{\overbrace{e^x \qquad dx}^{De^x}}{\sqrt{1-e^{2x}}} = \operatorname{arcsen} e^x + C$$

$$e^{2x} = (e^x)^2$$

$$\int \frac{e^{\operatorname{arctg} x}}{1+x^2}\,dx = e^{\operatorname{arctg} x} + c$$

$$D(\operatorname{arctg} x) = \frac{1}{1+x^2}$$

$$-\frac{1}{5}\int -5\,\operatorname{sen}(5x+4)\,dx = -\frac{1}{5}\cos(5x+4) + C$$

$$D(5x+4) = 5$$

$$\int \frac{1}{\sqrt{1-\ln^2 x}}\,\frac{1}{x}\,dx = \operatorname{arcsen}(\ln x) + c$$

$$\frac{1}{x}\,dx \downarrow D(\ln x)$$

$$\int (2x+5)\,\operatorname{sen}(x^2+5x)\,dx = -\cos(x^2+5x) + C$$

$$D(x^2+5x) = 2x+5$$

$$\int \frac{\overbrace{2}^{D(2x)}}{\sqrt{1-4x^2}}\,dx = \arcsin 2x + c$$

$4x^2 = (2x)^2$

$D(2x) = 2$

$$\int \sqrt{\frac{3+\operatorname{tg}x}{\cos^2 x}}\,dx = \frac{2}{3}(3+\operatorname{tg}x)^{3/2} + c$$

$D(\operatorname{tg}x) = \frac{1}{\cos^2 x}$

$\sqrt{3+\operatorname{tg}x} = (3+\operatorname{tg}x)^{1/2}$

$$\int e^x \operatorname{sen}(e^x)\,dx = -\cos(e^x) + c$$

$D(e^x) = e^x$ argomento sen

$$\int (3x^2+2)\cos(x^3+2x+1)\,dx =$$

$D(x^3+2x+1) = 3x^2+2$

$= \operatorname{sen}(x^3+2x+1) + c$

$$\int \frac{e^{\arcsin x}}{\sqrt{1-x^2}}\,dx = e^{\arcsin x} + c$$

$D(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$

T.E. data la funzione $f(x) = \frac{(1-\cos^2(2\log x))\cdot\cos(2\log(x))}{x}$

calcolare $F(e^{\pi/4}) = 1/4$

$$F(x) = \int \frac{\overbrace{(1-\cos^2(2\log x))}^{sen^2(2\log x)}\cdot\cos(2\log(x))}{x}\frac{2}{2}dx = \frac{1}{3}\cdot\frac{1}{2}\,sen^3(2\log x) + c =$$

$$= \frac{1}{6}\,sen^3(2\log x) + c$$

$$1-\cos^2(2\log x) = sen^2(2\log x)$$

$$D(sen(2\log x)) = \cos(2\log x)\cdot 2\cdot\frac{1}{x}$$

$F(e^{\frac{\pi}{4}}) = \frac{1}{4}$:

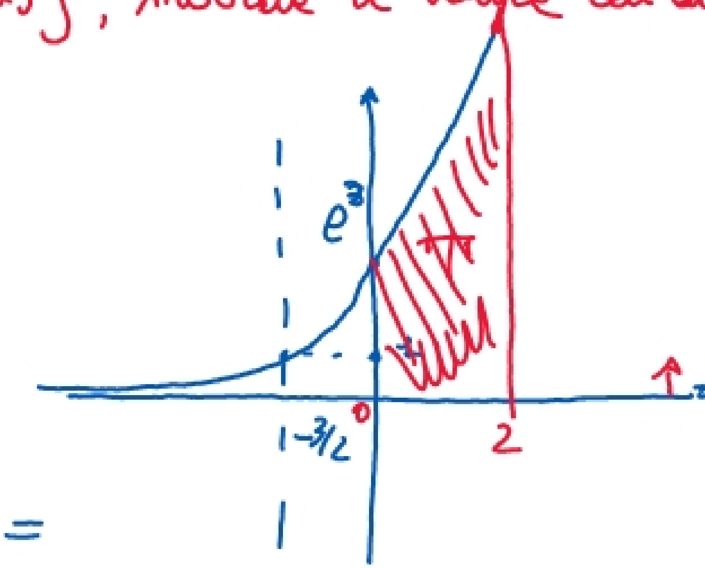
$$\frac{1}{6}\,sen^3\left(2\log e^{\frac{\pi}{4}}\right) + c = \frac{1}{4}$$

$$\frac{1}{6}\,\underbrace{sen^3\left(2\,\frac{\pi}{4_2}\,\cancel{\log e}_1\right)}_{1} + c = \frac{1}{4} \qquad c = \frac{1}{4} - \frac{1}{6} = \frac{1}{12}$$

$$\Rightarrow F(x) = \frac{1}{6}\,sen^3(2\log x) + \frac{1}{12}$$

T.E. data la funzione $f(x) = e^{x} \cdot e^{x+3}$

$T = \{(x, y) \in R^2 : 0 \le x \le 2,\ 0 \le y \le f(x)\}$, trovare il valore dell'area



$$f(x) = e^{x} \cdot e^{x+3} = e^{x+x+3} = e^{2x+3}$$

$$A_T = \int_0^2 \frac{2}{2} e^{2x+3}\, dx = \frac{1}{2}\left[e^{2x+3}\right]_0^2 =$$

$$D\left(e^{2x+3}\right) = e^{2x+3} \cdot \underbrace{2}_{\text{Der. esponente}}$$

$$= \frac{1}{2}\left[e^{2\cdot 2+3} - e^{0+3}\right] = \frac{1}{2}\left(e^{7} - e^{3}\right)$$

T.E. $\displaystyle\int_1^3 \frac{\log x}{x(9-\log^2 x)}\,dx =$

$$D(9-\log^2 x) = -2\log x \cdot \frac{1}{x}$$

$$D\left[\log\left|9-\log^2 x\right|\right] = \frac{1}{9-\log^2 x}\cdot(-2\log x)\cdot\frac{1}{x} = \frac{-2\log x}{x(9-\log^2 x)}$$

$$\int_1^3 \frac{-2}{-2}\cdot\frac{\log x}{x(9-\log^2 x)}\,dx = -\frac{1}{2}\left[\log\left|9-\log^2 x\right|\right]_1^3 =$$

$$= -\frac{1}{2}\left[\log\left|9-\log^2 3\right| - \log\left|9-\underbrace{\log^2 1}_{0}\right|\right] =$$

$$= -\frac{1}{2}\log\left|\frac{9-\log^2 3}{9}\right|$$

INTEGRALI PER FUNZIONI RAZIONALI FRATTE

$$\int \frac{N(x)}{D(x)}\, d(x)$$

1. verificare sempre che il grado di $N(x)$ sia inferiore al grado di $D(x)$

   altrimenti scomporre, dividere, ricorrere ad artifici per

   "semplificare" la frazione $\qquad \frac{N(x)}{D(x)} = q(x) + \frac{\varphi(x)}{D(x)}$

2. verificare che il denominatore sia scomponibile, in tal caso scomporlo:

$$f(x) = (x-a)^k f_1(x) \quad \text{t.c. } f_1(a) \neq 0$$

$x=a$ radice del denominatore di molteplicità k

$$\frac{F(x)}{f(x)} = Q(x) + \frac{\varphi(x)}{f(x)} \Rightarrow \frac{\varphi(x)}{(x-a)^k f_1(x)} = \frac{A}{(x-a)^k} + \frac{\varphi_1(x)}{(x-a)^k f_1(x)}$$

si deve tener conto del grado e delle molteplicità

3. se si ottiene un polinomio di 2° grado non scomponibile

rendelo, se possibile, del tipo:

a) $(t^2+1) \rightarrow \operatorname{arctg} t$

b) verificare se:

- il numeratore è la derivata del denominatore

$$\frac{2rx+q}{rx^2+qx+t} \rightarrow \ln|rx^2+qx+t|$$

SCOMPOSIZIONE DEL POLINOMIO FRAZIONARIO

- RADICI SEMPLICI

$$D(x) = (x-\alpha_1)\cdot(x-\alpha_2)\ldots\underset{\Delta<0}{(x^2+px+q)}$$

con $\alpha_1, \alpha_2 \ldots$ Radici per $D(x)$,

$$\Rightarrow \frac{N(x)}{D(x)} = \frac{A_1}{x-\alpha_1} + \frac{A_2}{x-\alpha_2} + \ldots + \frac{B_1x+C}{x^2+px+q}$$

- RADICI MULTIPLE

$$D(x) = (x-\alpha)^k \cdot (x^2+px+q)^s$$

$(x-\alpha)$: con molteplicità r

(x^2+px+q): con potenza s:

$$\frac{N(x)}{D(x)} = \frac{A_1}{x-\alpha} + \frac{A_2}{(x-\alpha)^2} + \ldots + \frac{A_r}{(x-\alpha)^r} +$$

$$+ \frac{B_1x+C_1}{(x^2+px+q)} + \frac{B_2x+C_2}{(x^2+px+q)^2} + \ldots + \frac{B_sx+C_s}{(x^2+px+q)^s}$$

es 1 $\int \frac{x+4}{x^2-5x+6}\,dx = \int \left[\frac{A}{x-3} + \frac{B}{x-2} \right] dx =$

$$x^2-5x+6 = (x-3)(x-2)$$

$$\frac{A}{x-3} + \frac{B}{x-2} = \frac{A(x-2)+B(x-3)}{(x-2)(x-3)} = \frac{Ax+Bx-2A-3B}{(x-2)(x-3)} =$$

$$\frac{x(A+B)-(2A+3B)}{(x-2)(x-3)} = \frac{x+4}{x^2-5x+6}$$

$$\begin{cases} A+B=1 \\ -(2A+3B)=4 \end{cases} \quad \begin{cases} A=1-B \\ 2(1-B)+3B=-4 \end{cases} \quad \begin{cases} A=1-B \\ 2\underbrace{-2B+3B}_{B}=-4 \end{cases} \quad \begin{cases} A=7 \\ B=-6 \end{cases}$$

$$= \int \left[\frac{7}{x-3} - \frac{6}{x-2} \right] dx = 7\ln|x-3| - 6\ln|x-2| + C$$

es 2

$$\int \frac{x^3+2}{x^3-x}\,dx = \int \frac{x^3-x+x+2}{x^3-x}\,dx = \int \left(1+\frac{x+2}{x^3-x}\right)dx =$$

$$= x + \int \left[\frac{A}{x}+\frac{B}{x-1}+\frac{C}{x+1}\right]dx = (*)$$

$x^3-x = x\,(x^2-1) = x\,(x-1)\,(x+1)$ 3 fattori di primo grado

$$\Rightarrow A(x-1)(x+1) + Bx(x+1) + Cx(x-1) = A(x^2-1) + Bx^2 + Bx + Cx^2 - Cx$$

$$\begin{array}{l} x^2 \\ x \\ x^0 \end{array}\begin{cases} A+B+C=0 \\ B-C=1 \\ -A=2 \end{cases} \qquad \begin{cases} A=-2 \\ B=+3/2 \\ C=+1/2 \end{cases}$$

$$(*) = x + \int\left[\frac{-2}{x} + \frac{3}{2}\,\frac{1}{x-1} + \frac{1}{2}\cdot\frac{1}{x+1}\right]dx =$$

$$= x - 2\ln|x| + \frac{3}{2}\ln|x-1| + \frac{1}{2}\ln|x+1| + c$$

es3. $\displaystyle\int \frac{2+3x+x^2}{x(x^2+1)}\,dx = \int \left[\frac{A}{x} + \frac{Bx+C}{x^2+1}\right] dx$

x , (x^2+1) non è scomponibile in $\mathbb{R}$

$\Rightarrow A(x^2+1) + x(Bx+C) = Ax^2 + A + Bx^2 + Cx$

$$\begin{array}{l} x^2 \\ x \\ x^0 \end{array}\begin{cases} A+B=1 \\ C=3 \\ A=2 \end{cases} \qquad \begin{cases} A=2 \\ B=1-A=1-2=-1 \\ C=3 \end{cases}$$

$$= \int \left[\frac{2}{x} + \frac{-2}{-2}\,\frac{-x+3}{x^2+1}\right] dx = 2\ln|x| + \underbrace{\int \frac{2x-6}{x^2+1}\left(-\frac{1}{2}\right) dx}_{(*)} = ⊛$$

$$(*) \int \left(\frac{2x}{x^2+1} - \frac{6}{x^2+1}\right)\left(-\frac{1}{2}\right) dx = -\frac{1}{2}\ln(x^2+1) + 3\,\mathrm{arctg}\,x + c_1$$

$$⊛ = 2\ln|x| - \frac{1}{2}\ln(x^2+1) + 3\,\mathrm{arctg}\,x + c$$

es 4. $\displaystyle \int \frac{1}{x^3+1}\,dx = \int \frac{1}{(x+1)(x^2-x+1)}\,dx = \int \left[\frac{A}{x+1} + \frac{Bx+C}{x^2-x+1} \right] dx$

$$A(x^2-x+1) + (Bx+C)(x+1) = \underline{Ax^2} - \underbrace{Ax} + A + \underline{Bx^2} + \underbrace{Cx} + \underbrace{Bx} + C$$

$$\begin{array}{l} x^2 \\ x \\ x^0 \end{array} \begin{cases} A+B=0 \\ -A+B+C=0 \\ A+C=1 \end{cases} \Longrightarrow \begin{cases} A = 1/3 \\ B = -1/3 \\ C = 2/3 \end{cases}$$

$$= \int \left[\frac{1}{3}\cdot\frac{1}{x+1} + \underbrace{\frac{-\frac{1}{3}x + 2/3}{x^2-x+1}}_{(*)} \right] dx = \frac{1}{3}\ln|x+1| \underbrace{- \frac{1}{6}\ln|x^2-x+1|}_{(*)} + \frac{1}{\sqrt{3}}\,\mathrm{arctg}\left(\frac{2}{\sqrt{3}}x - \frac{1}{\sqrt{3}}\right) + c$$

$$* \int \frac{-\frac{1}{3}x + 2/3}{x^2-x+1}\,dx = -\frac{1}{3}\int \frac{x+2}{x^2-x+1}\,dx = -\frac{1}{3}\int \frac{2x+4}{2(x^2-x+1)}\,dx =$$

$$= -\frac{1}{3}\cdot\frac{1}{2}\int \left(\frac{2x-1}{x^2-x+1} + \frac{5}{x^2-x+1} \right) dx = \underbrace{-\frac{1}{6}\ln|x^2-x+1|}_{(*)} + \int \frac{5}{x^2-x+1}\,dx$$

$$\int \frac{5}{x^2-x+1}\,dx = 5\int \frac{1}{\underbrace{x^2-x+1}}\,dx =$$

$$x^2-x+1 = \left(x^2-x+\frac{1}{4}\right)+\frac{3}{4} = \frac{3}{4}\left[\frac{\left(x-\frac{1}{2}\right)^2}{3/4}+1\right] =$$

rendere quadrato

$$= \frac{3}{4}\left[\frac{\left(x-\frac{1}{2}\right)^2}{\left(\sqrt{3}/2\right)^2}+1\right] = \frac{3}{4}\left[\left(\frac{x-\frac{1}{2}}{\sqrt{3}/2}\right)^2+1\right] =$$

$$= \frac{3}{4}\left[\left(\frac{2}{\sqrt{3}}x-\frac{1}{\sqrt{3}}\right)^2+1\right]$$

$$= 5\cdot\int \frac{1}{\frac{3}{4}\left[\left(\frac{2}{\sqrt{3}}x-\frac{1}{\sqrt{3}}\right)^2+1\right]}\,dx = \int \frac{5\cdot 4}{3}\,\frac{1}{\left[\left(\frac{2}{\sqrt{3}}x-\frac{1}{\sqrt{3}}\right)^2+1\right]}\,dx =$$

$$= \int \frac{5\cdot 2}{\sqrt{3}}\,\frac{2/\sqrt{3}}{\left(\frac{2}{\sqrt{3}}x-\frac{1}{\sqrt{3}}\right)^2+1}\,dx = \frac{10}{\sqrt{3}}\,\operatorname{arctg}\left(\frac{2}{\sqrt{3}}x-\frac{1}{\sqrt{3}}\right)$$

es 5. $\displaystyle\int \frac{1}{x(x-1)^2}\,dx = \int \left[\frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} \right] dx =$

fattore $(x-1)$ con molteplicità 2

(x^2-2x+1)

$A(x-1)^2 + Bx(x-1) + C = 1$

$$\begin{array}{l} x^2 \\ x \\ x^0 \end{array} \begin{cases} A+B=0 \\ -2A-B+C=0 \\ +A=1 \end{cases} \qquad \begin{cases} A=1 \\ B=-A=-1 \\ C=2A+B=2-1=1 \end{cases}$$

$$= \int \left[\frac{1}{x} - \frac{1}{x-1} + \frac{1}{(x-1)^2} \right] dx =$$

$$= \ln|x| - \ln|x-1| - \frac{1}{x-1} + c$$

es6 • $\displaystyle\int \frac{x^2+2}{x(2x^2+1)^2}\,dx = \int\left[\frac{A}{x} + \frac{Bx+C}{2x^2+1} + \frac{Dx+E}{(2x^2+1)^2}\right]dx$

$(2x^2+1)$ fattore di 2° grado con molteplicità 2

$$A(2x^2+1)^2 + (Bx+C)\,x\,(2x^2+1) + (Dx+E)\,x = x^2+2$$

$$\begin{array}{l} x^4 \\ x^3 \\ x^2 \\ x \\ x^0 \end{array} \begin{cases} 4A+2B=0 \\ 2C=0 \\ 4A+B+D=1 \\ C+E=0 \\ A=2 \end{cases} \Rightarrow \begin{cases} A=2 \\ B=-4 \\ C=0 \\ D=1-B-4A=1+4-8=-3 \\ E=0 \end{cases}$$

$$= \int\left[\frac{2}{x} + \frac{-4x}{2x^2+1} + \frac{-3\overset{4}{x}}{4(2x^2+1)^2}\right]dx =$$

$$= 2\ln|x| - \ln|2x^2+1| + 3\cdot\frac{1}{2x^2+1} + c$$

es 7

$$\int \frac{3x+1}{x^2-5x+6}\,dx = \int \left(\frac{A}{(x-2)} + \frac{B}{(x-3)} \right) dx =$$

$(x-2)(x-3)$

$$\frac{3x+1}{x^2-5x+6} \Longrightarrow \frac{A}{x-2} + \frac{B}{x-3} = \frac{x(A+B)-(3A+2B)}{(x-2)(x-3)}$$

uguaglianza polinomiale:

$$\begin{cases} A+B=3 \\ -3A-2B=1 \end{cases} \Longrightarrow \begin{cases} A=-7 \\ B=10 \end{cases}$$

$$= \int -\frac{7}{x-2}\,dx + \int \frac{10}{x-3}\,dx = -7\log|x-2| + 10\log|x-3| + c$$

es 8 $\displaystyle\int \frac{x+2}{x^2-4x+4}\,dx = \int\left[\frac{A}{x-2}+\frac{B}{(x-2)^2}\right]dx$

$(x-2)^2$

$\left(2^\circ \text{grado} \quad \Delta = 0\right)$

$\Rightarrow \dfrac{A(x-2)+B}{(x-2)^2} = \dfrac{Ax-2A+B}{(x-2)^2} \overset{\text{id. pol.}}{\Longrightarrow} \begin{cases} A = 1 \\ -2A+B = 2 \end{cases} \Rightarrow \begin{cases} A = 1 \\ B = 4 \end{cases} \ldots$

$\Rightarrow \displaystyle\int \frac{x+\overline{2-4}+4}{(x-2)^2}\,dx = \int \frac{x-2+4}{(x-2)^2}\,dx = \int\left(\frac{x-2}{(x-2)^2}+\frac{4}{(x-2)^2}\right)dx =$

$= \displaystyle\int \frac{1}{x-2}\,dx + 4\int \frac{1}{(x-2)^2}\,dx = \log|x-2| - \frac{4}{x-2} + c$

$D\left(\dfrac{1}{(x-2)}\right) = -\dfrac{1}{(x-2)^2}$

es. $\displaystyle\int \frac{x+1}{x^2+x+1}\,dx =$

$(\Delta<0)$

den. non è scomponibile

$$= \frac{1}{2}\int \frac{2(x+1)}{x^2+x+1}\,dx = \frac{1}{2}\int \frac{2x+2}{x^2+x+1}\,dx = \frac{1}{2}\int\left[\frac{2x+1}{x^2+x+1} + \underset{(*)}{\frac{1}{x^2+x+1}}\right]dx =$$

$$= \frac{1}{2}\log|x^2+x+1| + \frac{1}{2}\cdot\frac{2}{\sqrt{3}}\operatorname{arctg}\left(\frac{2x+1}{\sqrt{3}}\right) + c = \frac{1}{2}\log(x^2+x+1) + \frac{1}{\sqrt{3}}\operatorname{arctg}\left(\frac{2x+1}{\sqrt{3}}\right) + c$$

$\Downarrow$

$\log(x^2+x+1)$ perché l'argomento è una quantità sempre positiva

$$\frac{1}{3/4} = \frac{4}{3} = \frac{2}{\sqrt{3}}\cdot\frac{2}{\sqrt{3}}$$

$$= \int \frac{1}{x^2+x+1}\,dx = \int \frac{1}{\frac{3}{4}\left[\left(\frac{2x+1}{\sqrt{3}}\right)^2+1\right]}\,dx = \frac{2}{\sqrt{3}}\int \frac{\frac{2}{\sqrt{3}}}{\left(\frac{2x+1}{\sqrt{3}}\right)^2+1}\,dx = \frac{2}{\sqrt{3}}\operatorname{arctg}\left(\frac{2x+1}{\sqrt{3}}\right)+c$$

$$x^2+x+1 = \left(x^2+x+\frac{1}{4}\right)+\frac{3}{4} = \left(x+\frac{1}{2}\right)^2+\frac{3}{4} = \frac{3}{4}\left[\frac{\left(x+\frac{1}{2}\right)^2}{3/4}+1\right] = \frac{3}{4}\left[\left(\frac{2x+1}{\sqrt{3}}\right)^2+1\right]$$

$$\frac{\left(x+\frac{1}{2}\right)^2}{3/4} = \frac{(2x+1)^2}{\cancel{4}}\cdot\frac{\cancel{4}}{3} = \left(\frac{2x+1}{\sqrt{3}}\right)^2$$

es 10

$$\int \frac{1}{x^4-1}\,dx = \int \left[\frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1} \right] dx =$$

$$x^4-1 = (x^2-1)(x^2+1) = (x-1)(x+1)\underset{\Delta<0}{(x^2+1)}$$

$$\frac{A(x+1)(x^2+1) + B(x-1)(x^2+1) + (Cx+D)(x-1)(x+1)}{(x-1)(x+1)(x^2+1)} = \text{------} =$$

$$= \frac{(A+B+C)\,x^3 + (A-B+D)\,x^2 + (A+B-C)\,x + (A-B-D)}{(x-1)(x+1)(x^2+1)} = \frac{1}{x^4-1}$$

$$\begin{array}{l} x^3 \\ x^2 \\ x \\ x^0 \end{array}\begin{cases} A+B+C=0 \\ A-B+D=0 \\ A+B-C=0 \\ A-B-D=1 \end{cases} \Rightarrow \cdots \Rightarrow \begin{cases} A=1/4 \\ B=-1/4 \\ C=0 \\ D=-\frac{1}{2} \end{cases}$$

$$= \frac{1}{4}\int \frac{1}{x-1}\,dx - \frac{1}{4}\int \frac{1}{x+1}\,dx - \frac{1}{2}\int \frac{1}{x^2+1}\,dx =$$

$$= \frac{1}{4}\log|x-1| - \frac{1}{4}\log|x+1| - \frac{1}{2}\operatorname{arctg} x + C$$

es 11

$$\int \frac{1}{x^3(x-2)}\,dx = \int\left(\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-2}\right)dx =$$

$(x^3) \rightarrow \quad x=0$ con molteplicità 3

$(x-2) \rightarrow \quad x=2$ con molteplicità 1

$$\frac{Ax^2(x-2) + Bx(x-2) + C(x-2) + D(x^3)}{x^3(x-2)} = \frac{1}{x^3(x-2)} \quad \text{- - - -}$$

$$\begin{array}{l} x^3 \\ x^2 \\ x \\ x^0 \end{array}\begin{cases} A+D = 0 \\ -2A+B = 0 \\ -2B+C = 0 \\ -2C = 1 \end{cases} \Rightarrow \text{- -} \begin{cases} A = -1/8 \\ B = -1/4 \\ C = -1/2 \\ D = 1/8 \end{cases}$$

$$\int \frac{1}{x^2}\,dx = -\frac{1}{x} + c_1$$

$$\int \frac{1}{x^3}\,dx = \int x^{-3}\,dx = -\frac{1}{2}x^{-2} + c_2$$

$$= -\frac{1}{8}\int\frac{1}{x}\,dx - \frac{1}{4}\int\frac{1}{x^2}\,dx - \frac{1}{2}\int\frac{1}{x^3}\,dx + \frac{1}{8}\int\frac{1}{x-2}\,dx =$$

$$= -\frac{1}{8}\ln|x| + \frac{1}{4x} + \frac{1}{4x^2} + \frac{1}{8}\ln|x-2| + C$$

es 12

$$\int \frac{dx}{x^2(x^2+3)}\,dx =$$

$$x^2\,(x^2+3) \longrightarrow \frac{C}{x} + \frac{A}{x^2} + \frac{Bx+D}{x^2+3} =$$

$\downarrow$

radici multiple $\rightarrow 2$
$x=0$

$$= \frac{C(x^2+3)\,x + A(x^2+3) + (Bx+D)\cdot x^2}{x^2(x^2+3)} =$$

$$= \frac{Cx^3 + 3Cx + Ax^2 + 3A + Bx^3 + Dx^2}{x^2(x^2+3)} =$$

$$\begin{array}{l} x^3 \\ x^2 \\ x \\ x^0 \end{array}\begin{cases} C+B=0 \\ A+D=0 \\ 3C=0 \\ 3A=1 \end{cases} \Rightarrow \begin{cases} A = 1/3 \\ B = -C = 0 \\ C = 0 \\ D = -A = -1/3 \end{cases}$$

$$= \int \left(\cancel{\frac{0}{x}} + \frac{1}{3} \cdot \frac{1}{x^2} + \frac{\cancel{0x} - 1/3}{x^2+3} \right) dx =$$

$$= \int \left(\frac{1}{3} \frac{1}{x^2} - \frac{1}{3} \frac{1}{x^2+3} \right) dx = -\frac{1}{3x} - \frac{1}{3\sqrt{3}} \underset{(*)}{\operatorname{arctg}} \left(\frac{x}{\sqrt{3}} \right) + C$$

$$\int \frac{1}{x^2} dx = -\frac{1}{x}$$

$$\int \frac{1}{x^2+3} dx = \int \frac{1}{\left(\frac{x^2}{3} + 1 \right) \cdot 3} dx = \int \frac{1}{3} \frac{1}{\left(\frac{x}{\sqrt{3}} \right)^2 + 1} dx =$$

$$\int \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} \frac{1}{\left(\frac{x}{\sqrt{3}} \right)^2 + 1} dx = \frac{1}{\sqrt{3}} \cdot \operatorname{arctg} \left(\frac{x}{\sqrt{3}} \right) + C_1 \ (*)$$

es 13

$$\int \frac{dx}{x^3-1} = \int \left(\frac{A}{x-1} + \frac{Bx+C}{x^2+x+1} \right) dx =$$

$$x^3-1 = (x-1)\underbrace{(x^2+x+1)}_{\text{fattore quadratico} \to \text{non ho sol. reali}}$$

$$A(x^2+x+1) + (Bx+C)(x-1) = 1$$

$$\begin{array}{l} x^2 \\ x \\ x^0 \end{array} \begin{cases} A + B = 0 \\ A - B + C = 0 \\ A - C = 1 \end{cases} \Rightarrow \begin{cases} A = 1/3 \\ B = -A = -1/3 \\ C = -1 + A = -1 + \frac{1}{3} = -2/3 \end{cases}$$

$$2(x+2) = 2x+4 \qquad 1+3$$

$$= \int \left[\frac{1/3}{x-1} + \frac{-1/3 x - 2/3}{x^2+x+1} \right] dx = \frac{1}{3} \ln|x-1| - \frac{1}{3} \underbrace{\int \frac{x+2}{x^2+x+1} \cdot \frac{1}{2}\, dx}_{(*)} =$$

$$(*) \int \frac{1}{2} \left(\frac{2x+1}{x^2+x+1} + \frac{3}{x^2+x+1} \right) dx = \frac{1}{2} \ln|x^2+x+1| + \frac{1}{2} \underbrace{\int \frac{3}{x^2+x+1}\, dx}_{(**)} =$$

$$(**) = \int \frac{3}{x^2+x+1}\, dx = \frac{1}{\sqrt{3}} \operatorname{arctg} \frac{2x+1}{\sqrt{3}} + C_1$$

con $x^2+x+1 = \left(x^2+x+\frac{1}{4}\right) + \frac{3}{4} = \cdot$

$$= \left(x^2 + x + \frac{1}{4}\right) + \frac{3}{4} = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4} = \frac{3}{4}\left[\frac{\left(x+\frac{1}{2}\right)^2}{3/4} + 1\right] =$$

$$= \frac{3}{4}\left[\left(2\,\frac{2x+1}{\sqrt{3}}\right)^2 \cdot \frac{1}{4} + 1\right] =$$

$$= \frac{3}{4}\left[\left(2\,\frac{2x+1}{\sqrt{3}} \cdot \frac{1}{2}\right)^2 + 1\right] = \frac{3}{4}\left[\left(\frac{2x+1}{\sqrt{3}}\right)^2 + 1\right]$$

L'integrale è:

$$= \frac{1}{3} \ln|x-1| - \frac{1}{6} \log(x^2+x+1) + \frac{1}{\sqrt{3}} \operatorname{arctg} \frac{2x+1}{\sqrt{3}} + C$$

es 14 $\displaystyle\int_{-2}^{1} \frac{t^2+3t+4}{t^2+4t+5}\,dt = \int_{-2}^{1}\Big(1 - \underbrace{\frac{t+1}{t^2+4t+5}}_{(*)}\Big)\,dt$

$\Delta < 0$

$t^2+4t+5 = 0 \qquad t_{1,2} = \dfrac{-2 \pm \sqrt{4-5}}{1}$ non sol. reali

$\Downarrow$

non è scomponibile

$$\frac{t^2+3t+4}{t^2+4t+5} = \frac{t^2 + \overbrace{4t - t}^{3t} + \overbrace{5 - 1}^{4}}{t^2+4t+5} =$$

$$= \frac{t^2+4t+5}{t^2+4t+5} - \frac{t+1}{t^2+4t+5} = 1 - \frac{t+1}{t^2+4t+5}$$

(N.B. si esegue la divisione polinomiale)

$$(*) \int \frac{t+1}{t^2+4t+5}\,dt = \int \frac{1}{2}\,\frac{2t+\overset{\downarrow 4-2}{2}}{t^2+4t+5}\,dt = \int \frac{1}{2}\left(\frac{2t+4}{t^2+4t+5} - \frac{2}{t^2+4t+5}\right)dt$$

$$= \frac{1}{2}\ln|t^2+4t+5| - \int \frac{1}{t^2+4t+5}\,dt$$

$$\int \frac{1}{t^2+4t+5}\,dt = \int \frac{1}{(t+2)^2+1}\,dt = \operatorname{arctg}(t+2) + C$$

$$t^2+4t+5 = (t^2+4t+4)+1 = (t+2)^2+1$$

$$D(t+2) = 1$$

$$= \int_{-2}^{1} \left(1 - \frac{t+1}{t^2+4t+5}\right) dt = x - \frac{1}{2}\ln(t^2+4t+5) + \operatorname{arctg}(t+2) \Big|_{-2}^{1} =$$

$$= \underbrace{1} - \frac{1}{2}\ln(1+4+5) + \operatorname{arctg}(3) \underbrace{+2} + \underbrace{\frac{1}{2}\ln(\underbrace{4-8+5}_{1})}_{0} - \underbrace{\operatorname{arctg}(0)}_{0} =$$

$$= 3 - \frac{1}{2}\ln(10) + \operatorname{arctg}(3)$$

es 15

$$\int \frac{2x-3}{x^2-x+2}\,dx = \int\left(\frac{2x-1}{x^2-x+2} - \frac{2}{x^2-x+2}\right)dx =$$

$$\downarrow$$

D (arctg(t))

$x^2-x+2=0$ $\quad \Delta<0$

$x_{1,2} = \frac{1 \pm \sqrt{1-8}}{2}$ non è scomponibile

$$= \ln(x^2-x+2) - 2\int \frac{1 \cdot \frac{2}{\sqrt{7}}}{\frac{2}{\sqrt{7}}(x^2-x+2)}\,dx =$$

$$\left(x^2-x+2 = x^2-x+\frac{1}{4}+\frac{7}{4} = \left(x-\frac{1}{2}\right)^2+\frac{7}{4} = \frac{7}{4}\left(\frac{\left(x-\frac{1}{2}\right)^2}{\frac{7}{4}}+1\right) = \right.$$

$$= \frac{7}{4}\left[\left(\frac{2\cdot\frac{2x-1}{2}}{\sqrt{7}}\right)^2+1\right) = \frac{7}{4}\left[\left(\frac{2}{\sqrt{7}}x-\frac{1}{\sqrt{7}}\right)^2+1\right] =$$

$$= \ln(x^2-x+2) - \frac{1}{\sqrt{7}}\,\mathrm{arctg}\left(\frac{2}{\sqrt{7}}x-\frac{1}{\sqrt{7}}\right) + C$$

es 16

$$\int \frac{x^3 - 2x^2 + 3}{x^3(x-1)}\,dx = \int\left(\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-1}\right)dx =$$

$x=0$ ha molteplicità 3

$$A x^2 + Bx + C + D x^3 = x^3 - 2x + 3$$

$$\begin{array}{l} x^3 \\ x^2 \\ x \\ x^0 \end{array} \begin{cases} D = 1 \\ A = 0 \\ B = -2 \\ C = 3 \end{cases}$$

$$= \int\left(\frac{0}{x} + \frac{-2}{x^2} + \frac{3}{x^3} + \frac{1}{x-1}\right)dx =$$

$$= + \frac{2}{x} - \frac{3}{2}\,\frac{1}{x^2} + \ln|x-1| + c$$

es 17

$$\int \frac{2x+10}{(x-2)(x^2+x+1)}\,dx = \int \left(\frac{A}{x-2} + \frac{Bx+C}{\underset{\Delta<0}{x^2+x+1}} \right) dx =$$

$x^2+x+1=0 \qquad x_{1,2} = \dfrac{-1 \pm \sqrt{1-4}}{2}$ non ha sol. reali, non è scomponibile

$$A(x^2+x+1) + (Bx+C)(x-2) = 2x+10$$

$$\begin{array}{l} x^2 \\ x \\ x^0 \end{array} \begin{cases} A+B=0 \\ A+C-2B=2 \\ A-2C=10 \end{cases} \Rightarrow \begin{cases} A=2 \\ B=-A=-2 \\ C=\dfrac{-10+A}{2} = \dfrac{-10+2}{2} = -8/2 = -4 \end{cases}$$

$$= \int \left[\frac{2}{x-2} + \frac{\overset{-(2x+4)}{-2x-4}}{x^2+x+1} \right] dx = \int \left[\frac{2}{x-2} - \frac{2x+1}{x^2+x+1} - \frac{3}{x^2+x+1} \right] dx =$$

$$= 2\ln|x-2| - \ln(x^2+x+1) - \sqrt{3}\cdot 2 \,\mathrm{arctg}\, \underbrace{\frac{x+\frac{1}{2}}{\sqrt{3}/2}}_{\left(\frac{2x+1}{\sqrt{3}}\right)} + c$$

T.E. $\displaystyle\int_0^{\pi/4} \frac{1-\mathrm{Tg}\,x}{1+\mathrm{Tg}\,x}\,dx =$

$$\frac{1-\mathrm{Tg}\,x}{1+\mathrm{Tg}\,x} = \frac{1-\dfrac{\mathrm{sen}\,x}{\cos x}}{1+\dfrac{\mathrm{sen}\,x}{\cos x}} = \frac{\cos x-\mathrm{sen}\,x}{\cos x}\cdot\frac{\cos x}{\cos x+\mathrm{sen}\,x} =$$

$$= \frac{\cos x-\mathrm{sen}\,x}{\cos x+\mathrm{sen}\,x}\cdot\frac{\cos x+\mathrm{sen}\,x}{\cos x+\mathrm{sen}\,x} = \frac{\cos^2 x-\mathrm{sen}^2 x}{\underbrace{\cos^2 x+\mathrm{sen}^2 x}_{1}+2\,\mathrm{sen}\,x\cos x} =$$

$$= \frac{\cos 2x}{1+\mathrm{sen}\,2x} \qquad D(1+\mathrm{sen}\,2x) = \cos 2x\cdot 2$$

$$= \int_0^{\pi/4} \frac{1}{2}\,\frac{2\cos 2x}{(1+\mathrm{sen}\,2x)}\,dx = \frac{1}{2}\log\left|1+\mathrm{sen}\,2x\right|\Big|_0^{\pi/4} =$$

$$= \frac{1}{2}\left(\log\underbrace{\left|1+\underbrace{\mathrm{sen}\,2\cdot\tfrac{\pi}{4_2}}_{1}\right|}_{2} - \log\underbrace{\left|1+\underbrace{\mathrm{sen}\,2\cdot 0}_{0}\right|}_{0}\right) = \frac{1}{2}\log 2$$