- SERIE -

$$\sum_{k=0}^{+\infty} a_k = a_0 + a_1 + \dots + a_n + \dots$$

somme parziali $\{S_n\}$

$S_0 = a_0$

$S_1 = S_0 + a_1 = a_0 + a_1$

$S_2 = S_1 + a_2 = (a_0 + a_1) + a_2 = a_0 + a_1 + a_2$

$\lim_{n \to +\infty} S_n = S$ carattere delle serie

$S = +\infty$ diverge positivamente

$S = -\infty$ diverge negativamente

$S \in]-\infty, +\infty[$ converge

$\nexists$ oscillante. non classificabile

S_n = successione delle somme parziali es 1

$$\sum_{n=1}^{+\infty} n$$

$S_1 = 1$

$S_2 = 1+2 = 3$

$S_3 = 1+2+3 = 6$

somme dei numeri naturali

calcolabile: $\longrightarrow S_n = 1+2+3+\dots+n = \dfrac{n(n+1)}{2}$ (dimostrazione pagine seguente)

$$\lim_{n\to+\infty} \frac{n(n+1)}{2} = +\infty$$

divergente positivamente

es 2

$$\sum_{n=2}^{+\infty} \frac{1}{n^2-1}$$

$S_2 = \dfrac{1}{3}$

$S_3 = \dfrac{1}{3} + \dfrac{1}{8}$

...

$S_n = \dfrac{1}{3} + \dfrac{1}{8} + \dfrac{1}{15} + \dots + \dfrac{1}{n^2-1}$

N.B.
attenzione agli indici delle serie

$$\frac{1}{n^2-1} = \frac{A}{n-1} + \frac{B}{n+1} = \frac{A(n+1)+B(n-1)}{(n-1)(n+1)} = \frac{An+A+Bn-B}{(n-1)(n+1} =$$

$n^2-1 = (n-1)(n+1)$

dimostrazione es 1

- data la successione $\{a_n\}$, $a_n = n$

dimostrare che $S_n = 1+2+3+\dots+n = \frac{n(n+1)}{2}$

[la somma dei primi n numeri è il semiprodotto del numero per il suo successivo]

$S_1 = 1 \qquad = \frac{1\cdot 2}{2} = 1$

$S_2 = 1+2=3 \qquad = \frac{2\cdot 3}{2} = 3$

$S_3 = 1+2+3=6 \qquad = \frac{3\cdot 4}{2} = 6$

per induzione:

$$S_n = \frac{n(n+1)}{2}^{(*)} \Rightarrow S_{n+1} = \frac{(n+1)(n+2)}{2}$$

si consideri $S_{n+1} = \underbrace{S_n + \overset{a_{n+1}}{(n+1)}} = \frac{\overset{(*)}{n(n+1)}}{2} + \overset{a_{n+1}}{(n+1)} =$

$= (n+1)\cdot\left(\frac{n}{2}+1\right) = \frac{(n+1)(n+2)}{2}$ c.v.d.

completamento es 2

$$\frac{n(A+B)+(A-B)}{(n-1)(n+1)} = \frac{1}{n^2-1} = \frac{A}{n-1} + \frac{B}{n+1} = \frac{A(n+1)+B(n-1)}{(n-1)(n+1)}$$

$$\begin{cases} A+B=0 \\ A-B=1 \end{cases} \qquad \begin{cases} 2A=1 \\ B=\cancel{A}-1 \end{cases} \qquad \begin{cases} A=\frac{1}{2} \\ B=-1+\frac{1}{2}=\frac{-1}{2} \end{cases}$$

$$\sum_{n=2}^{+\infty} \frac{1}{n^2-1} = \sum_{n=2}^{+\infty}\left(\frac{1/2}{n-1} + \frac{-\frac{1}{2}}{n+1}\right) = \frac{1}{2}\sum_{n=2}^{+\infty}\left(\frac{1}{n-1}-\frac{1}{n+1}\right)$$

$$S_n = \frac{1}{2}\left[\underset{n=2}{\left(1-\frac{1}{3}\right)} + \underset{n=3}{\left(\frac{1}{2}-\frac{1}{4}\right)} + \underset{n=4}{\left(\frac{1}{3}-\frac{1}{5}\right)} + \cdots + \underset{n=n-1}{\left(\frac{1}{n-2}-\frac{1}{n}\right)} + \underset{n=n}{\left(\frac{1}{n-1}-\frac{1}{n+1}\right)}\right] =$$

(denominatore: · precedente di n
· successivo di n)

$$= \frac{1}{2}\left[\frac{3}{2}-\frac{1}{n}-\frac{1}{n+1}\right]$$

$$\lim_{n\to+\infty} S_n = \lim_{n\to+\infty}\frac{1}{2}\left[\frac{3}{2}-\overset{\to 0}{\frac{1}{n}}-\overset{\to 0}{\frac{1}{n+1}}\right] = \frac{1}{2}\cdot\frac{3}{2} = \frac{3}{4} \quad \text{conv.}$$

$$\frac{1}{2}\sum_{n=2}^{+\infty}\left(\frac{1}{n-1}-\frac{1}{n+1}\right)$$

verifica dell'espressione sopra scritta per il calcolo della serie -

calcolo di

$$S_{10} = \frac{1}{2}\left(\underbrace{\frac{1}{1}-\frac{1}{3}}_{a_2}+\underbrace{\frac{1}{2}-\frac{1}{4}}_{a_3}+\underbrace{\frac{1}{3}-\frac{1}{5}}_{a_4}+\underbrace{\frac{1}{4}-\frac{1}{6}}_{a_5}+\underbrace{\frac{1}{5}-\frac{1}{7}}_{a_6}+\underbrace{\frac{1}{6}-\frac{1}{8}}_{a_7}+\underbrace{\frac{1}{7}-\frac{1}{9}}_{a_8}+\underbrace{\frac{1}{8}-\frac{1}{10}}_{a_9}+\underbrace{\frac{1}{9}-\frac{1}{11}}_{a_{10}}\right) =$$

$(n = 2 \dots 10)$

• termini che si elidono

$$= \frac{1}{2}\left(1+\frac{1}{2}-\frac{1}{10}-\frac{1}{11}\right) =$$

$$= \frac{1}{2}\left(\frac{3}{2}-\frac{1}{10}-\frac{1}{11}\right) \Rightarrow \text{in generale le } S_n = \frac{1}{2}\left(\frac{3}{2}-\frac{1}{n}-\frac{1}{n+1}\right)$$

cvd.

es3 $\sum_{n=1}^{+\infty} (-1)^{n+1} = 1 - 1 + 1 - 1 - - - + (-1)^{n+1} + - - -$

$$\{S_n\} = \begin{cases} 1 & n \text{ dispari} \\ 0 & n \text{ pari} \end{cases}$$

$\Rightarrow$ • l'insieme numerico ha 2 punti di accumulazione $\{0, 1\}$ (punti isolati)

• la successione è indeterminata

• la serie è oscillante

N.B. $\lim_{n \to +\infty} S_n = \nexists$

• SERIE TELESCOPICHE

$$\sum_{n=0}^{+\infty} a_n \quad \text{se} \quad a_n = b_n - b_{n+1}$$

$$\Rightarrow \quad S_n = \sum_{k=0}^{n} a_k = \sum_{k=0}^{n} \left(b_k - b_{k+1} \right) =$$

$$= \underbrace{b_0 - \cancel{b_1}}_{k=0} + \underbrace{\cancel{b_1} - b_2}_{k=1} + \dots - \cancel{b_n} + \underbrace{\cancel{b_n} - b_{n+1}}_{k=n} = b_0 - b_{n+1}$$

si semplificano gli elementi

$$S_n = b_0 - b_{n+1}$$

$$\lim_{n\to+\infty} S_n = \lim_{n\to+\infty} \left(b_0 - b_{n+1} \right) = b_0 - \lim_{n\to+\infty} b_{n+1} \quad (*)$$

$$(*) \begin{cases} < +\infty \ (\text{valore finito}) & \Rightarrow \text{la serie conv.} \\ +\infty \\ -\infty & \Rightarrow \text{la serie diverge} \\ \nexists & \Rightarrow \text{la serie può oscillare.} \end{cases}$$

es 1

$$\sum_{n=1}^{+\infty}\left(\underbrace{\frac{1}{n!}}_{b_n}-\underbrace{\frac{1}{(n+1)!}}_{b_{n+1}}\right)=\sum_{n=1}^{+\infty}\left(\frac{(n+1)!-n!}{n!\,(n+1)!}\right)=\sum_{n=1}^{+\infty}\frac{(n+1)\,n!-n!}{n!\,(n+1)!}=$$

$$=\sum_{n=1}^{+\infty}\frac{n!\,[n+1-1]}{n!\,(n+1)!}=$$

$$S_n=\sum_{k=1}^{n}a_k=b_1-b_{n+1}\qquad\qquad=\sum_{n=1}^{+\infty}\frac{n}{(n+1)!}$$

$$\lim_{n\to+\infty}S_n=\lim_{n\to+\infty}(b_1-b_{n+1})=\frac{1}{1!}-\underbrace{\lim_{n\to+\infty}\frac{1}{(n+1)!}}_{0}=1\quad \text{conv.}$$

es 2

$$\sum_{n=1}^{+\infty}\log\left(1+\frac{1}{n}\right)=\sum_{n=1}^{+\infty}\log\left(\frac{n+1}{n}\right)=$$

$$=\sum_{n=1}^{+\infty}\left[\log(n+1)-\log n\right]=-\sum_{n=1}^{+\infty}\left[\underbrace{\log n}_{b_n}-\underbrace{\log(n+1)}_{b_{n+1}}\right]$$

Termine iniziale   Termine (n+1)esimo

$$-S_n=b_1-b_{n+1}=\log 1-\log(n+1)=-\log(n+1)$$

$$\lim_{n\to+\infty}\left(-\left(-\log(n+1)\right)\right)=+\infty\qquad \text{div. pos.}$$

es 3 $\displaystyle\sum_{n=3}^{+\infty} \frac{(n+1)\,\mathrm{sen}(n\frac{\pi}{2}) - n\cos(n\frac{\pi}{2})}{n(n+1)} =$

$$= \sum_{n=3}^{+\infty}\left[\frac{\cancel{(n+1)}\,\mathrm{sen}(n\frac{\pi}{2})}{n\,\cancel{(n+1)}} - \frac{\cancel{n}\cos(n\frac{\pi}{2})}{\cancel{n}\,(n+1)}\right] =$$

$$\cos\left(n\frac{\pi}{2}\right) = \mathrm{sen}\left(n\frac{\pi}{2} + \frac{\pi}{2}\right) = \mathrm{sen}\left((n+1)\frac{\pi}{2}\right)$$

$$\left[\text{N.B. } \mathrm{sen}\left(x + \overset{90°}{\frac{\pi}{2}}\right) = \cos x\right]$$

$$= \sum_{n=3}^{+\infty}\left[\underbrace{\frac{\sin(n\frac{\pi}{2})}{n}}_{b_n} - \underbrace{\frac{\mathrm{sen}((n+1)\frac{\pi}{2})}{n+1}}_{b_{n+1}}\right]$$

$$s_n = b_3 - b_{n+1} = \frac{\mathrm{sen}(3\frac{\pi}{2})}{3} - \frac{\mathrm{sen}((n+1)\frac{\pi}{2})}{n+1}$$

$$\lim_{n\to+\infty} s_n = b_3 - \lim_{n\to+\infty} b_{n+1} = -\frac{1}{3} - \underbrace{\lim_{n\to+\infty} \frac{\overset{[-1,1]}{\mathrm{sen}((n+1)\frac{\pi}{2})}}{\underset{n\to+\infty}{n+1}}}_{0}$$

es 4 $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{n^2+2n} = \sum_{n=1}^{+\infty} \frac{1}{n(n+2)} =$

$$\frac{1}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2} = \frac{A(n+2)+Bn}{n(n+2)}$$

$$\begin{cases} A+B=0 \\ 2A=1 \end{cases} \qquad \begin{cases} B=-A=-\frac{1}{2} \\ A=\frac{1}{2} \end{cases}$$

$$= \sum_{n=1}^{+\infty} \left[\frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+2}\right)\right] =$$

$$= \frac{1}{2} \sum_{n=1}^{+\infty} \left(\frac{1}{n} - \frac{1}{n+2}\right) = \frac{1}{2}\sum_{n=1}^{+\infty} (b_n - b_{n+2})$$

$$S_n = \frac{1}{2}\left[\sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{k+2}\right)\right] = \frac{1}{2}\left[1 - \cancel{\frac{1}{3}} + \frac{1}{2} - \frac{1}{4} + \cancel{\frac{1}{3}} - \frac{1}{5} + \dots + \right.$$

$$\left. + \frac{1}{k-1} + \frac{1}{k+1} + \frac{1}{k} - \frac{1}{k+2}\right] =$$

$$= 1 + \frac{1}{2} - \frac{1}{k+1} - \frac{1}{k+2}$$

$$\lim_{n\to+\infty} S_n = \lim_{n\to+\infty} \frac{1}{2}\Big(\overbrace{1+\frac{1}{2}}^{3/2} - \underbrace{\frac{1}{k+1}}_{\to 0} + \underbrace{\frac{1}{k+2}}_{\to 0}\Big) = 3/4$$

N.B.: denominatore : 1 frazione è n

1 frazione è n+2 (successivo del successivo)

$$\frac{1}{2}\sum_{n=1}^{+\infty}\left(\frac{1}{n}-\frac{1}{n+2}\right)$$

Calcolo di S_5

$$S_5 = \frac{1}{2}\left(\underbrace{1-\frac{1}{3}}_{a_1}+\underbrace{\frac{1}{2}-\frac{1}{4}}_{a_2}+\underbrace{\frac{1}{3}-\frac{1}{5}}_{a_3}+\underbrace{\frac{1}{4}-\frac{1}{6}}_{a_4}+\underbrace{\frac{1}{5}-\frac{1}{7}}_{a_5}\right)=$$

• Termini che si elidono

$$=\frac{1}{2}\left(1+\frac{1}{2}-\frac{1}{6}-\frac{1}{7}\right)=\frac{1}{2}\left(\frac{3}{2}-\underset{5+1}{\frac{1}{6}}-\underset{5+2}{\frac{1}{7}}\right)$$

quindi in generale $S_n = \frac{1}{2}\left(\frac{3}{2}-\frac{1}{n+1}-\frac{1}{n+2}\right)$

cvd

SERIE GEOMETRICHE

$$\sum_{n=0}^{+\infty} a\, q^n = a \cdot \sum_{n=0}^{+\infty} q^n$$

$$a \neq 0$$

$$\begin{cases} |q| < 1 & \text{convergente} \\ q \geq 1 & \text{divergente} \\ q = -1 & \text{indeterminata oscill.} \\ q < -1 & \text{indeterminata oscill.} \end{cases}$$

es 1

$$\sum_{n=0}^{+\infty} \left(3^{-n} \left(\underbrace{\sum_{k=0}^{n-1} \left(\frac{3}{2}\right)^k}_{S_n} \right) \right) =$$

$$\sum_{k=0}^{n-1} \left(\frac{3}{2}\right)^k = \frac{1-\left(\frac{3}{2}\right)^n}{1-3/2}$$

$$q = \frac{3}{2} > 1$$

$$\boxed{S_n = \frac{1-q^{n+1}}{1-q}}$$

$$= \sum_{n=0}^{+\infty} \left(3^{-n} \cdot \frac{1-\left(\frac{3}{2}\right)^n}{\underbrace{1-3/2}_{-1/2}} \right) =$$

$$= -2 \sum_{n=0}^{+\infty} \left(3^{-n} - \left(\frac{3}{2}\right)^n \cdot 3^{-n} \right) = -2 \left(\sum_{n=0}^{+\infty} \left(\left(\frac{1}{3}\right)^n - \left(\frac{1}{2}\right)^n \right) \right) =$$

$$= -2 \left(\sum_{n=0}^{+\infty} \left(\frac{1}{3}\right)^n - \sum_{n=0}^{+\infty} \left(\frac{1}{2}\right)^n \right) = -2 \left(\frac{1}{1-\frac{1}{3}} - \frac{1}{1-\frac{1}{2}} \right) = 1$$

$$\boxed{\frac{1}{3}, \frac{1}{2} < 1}$$

$$q < 1 \qquad \frac{1}{1-q}$$

SCHEMA SERIE GEOMETRICA

$$\sum_{n=0}^{+\infty} q^n \quad \text{se}$$

$\lvert q \rvert < 1$	convergenza
$q > 1$	divergenza
$q = 1$	divergenza
$q = -1$	indeterminata
$q < -1$	indeterminata

s_n = le somme parziali :

$|q| < 1 \qquad s_n = \dfrac{1}{1-q}$

$|q| > 1 \qquad s_n = \dfrac{1-q^n}{1-q} = \dfrac{q^n - 1}{q-1}$ (cambio di segno)

es 2

$$a_n = \begin{cases} 7^{-\frac{n}{2}} = \left[\left(\frac{1}{7}\right)^n\right]^{1/2} & n \text{ pari} \\ -(7)^{-n} = -\left(\frac{1}{7}\right)^n & n \text{ dispari} \end{cases}$$

calcolare $\sum_{n=1}^{+\infty} a_n =$

schema valori

n:	1	2	3	4	5	6	7
n pari		$\left(\frac{1}{7}\right)^{\frac{2}{2}=1}$		$\left(\frac{1}{7}\right)^{\frac{4}{2}=2}$		$\left(\frac{1}{7}\right)^{\frac{6}{2}=3}$	
n dispari	$-\left(\frac{1}{7}\right)^1$		$-\left(\frac{1}{7}\right)^3$		$-\left(\frac{1}{7}\right)^5$		$-\left(\frac{1}{7}\right)^7$

$$\sum_{n=1}^{+\infty} a_n = \underbrace{\sum_{n \text{ pari}} a_n}_{\text{valore positivo}} + \underbrace{\sum_{n \text{ dispari}} a_n}_{\text{valore negativo}} = \sum_{n \text{ pari}} \left(\frac{1}{7}\right)^{\frac{n}{2}} + \sum_{n \text{ dispari}} -\left(\frac{1}{7}\right)^n =$$

$$= \sum_{k=1}^{+\infty} \left(\frac{1}{7}\right)^k - \sum_{k=0}^{+\infty} \left(\frac{1}{7}\right)^{2k+1} =$$

2 serie geometriche di ragione $q = \frac{1}{7} < 1$

$$= \sum_{k=1}^{+\infty} \left(\frac{1}{7}\right)^k - \frac{1}{7} \sum_{k=0}^{+\infty} \left[\left(\frac{1}{7}\right)^2\right]^k = \sum_{k=1}^{+\infty} \left(\frac{1}{7}\right)^k - \frac{1}{7} \sum_{k=0}^{+\infty} \left(\frac{1}{49}\right)^k =$$

non parte da 0 (*)

$$= \sum_{k=0}^{+\infty} \left(\frac{1}{7}\right)^k \underbrace{-1}_{(*)\ k=0} - \frac{1}{7} \sum_{k=0}^{+\infty} \left(\frac{1}{49}\right)^k = \frac{1}{1-\frac{1}{7}} - 1 - \frac{1}{7} \cdot \frac{1}{1-\frac{1}{49}} = \frac{7}{6} - 1 - \frac{1}{\cancel{7}_1} \cdot \frac{\cancel{49}^{7}}{48}$$

$$= \frac{56-48-7}{48} = \frac{1}{48}$$

es 3 Sia $b \in \mathbb{R}$, la serie $\displaystyle\sum_{n=0}^{+\infty} \frac{\cosh(b \cdot n)}{2^n}$ converge per $b = ?$

$$\cosh(bn) = \frac{e^{bn} + e^{-bn}}{2}$$ serie a T. p.

$$\sum_{n=0}^{+\infty} \frac{\cosh(bn)}{2^n} = \sum_{n=0}^{+\infty} \frac{e^{bn} + e^{-bn}}{2 \cdot 2^n} = \sum_{n=0}^{+\infty} \frac{e^{bn}}{2 \cdot 2^n} + \sum_{n=0}^{+\infty} \frac{e^{-bn}}{2 \cdot 2^n} =$$

$$= \frac{1}{2} \sum_{n=0}^{+\infty} \left(\frac{e^{b}}{2}\right)^n + \frac{1}{2} \sum_{n=0}^{+\infty} \left(\frac{e^{-b}}{2}\right)^n$$ serie geometriche

conv. $|q| < 1$

$$\begin{cases} \dfrac{e^{b}}{2} < 1 \\ \dfrac{e^{-b}}{2} < 1 \end{cases} \Rightarrow \begin{cases} e^{b} < 2 \\ e^{-b} < 2 \end{cases} \quad \begin{cases} b < \log 2 \\ -b < \log 2 \end{cases} \quad -\log 2 < b < \log 2$$

$$\Rightarrow |b| < \log 2$$

Calcolo del valore delle serie:

es 4

$$\sum_{n=1}^{+\infty}\left(\frac{2}{5}\right)^n = \sum_{n=0}^{+\infty}\left(\frac{2}{5}\right)^n - a_0 = \sum_{n=0}^{+\infty}\left(\frac{2}{5}\right)^n - 1 =$$

$$= \frac{1}{1-\frac{2}{5}} - 1 = \frac{5}{3} - 1 = \frac{2}{3}$$

serie geometrica $|q|<1$

es 5:

$$\sum_{n=1}^{+\infty} 3^{-n} = \sum_{n=1}^{+\infty}\left(\frac{1}{3}\right)^n = \sum_{n=0}^{+\infty}\left(\frac{1}{3}\right) - \underbrace{1}_{a_0} =$$

$$= \frac{1}{1-\frac{1}{3}} - 1 = \frac{3}{2} - 1 = \frac{1}{2}$$

es 6

$$\sum_{n=0}^{+\infty}\frac{5}{2\cdot 2^n} = \frac{5}{2}\sum_{n=0}^{+\infty}\left(\frac{1}{2}\right)^n = \frac{5}{2}\,\frac{1}{1-\frac{1}{2}} = \frac{5}{2}\cdot 2 = 5$$

es 7

$$\sum_{n=0}^{+\infty}\frac{2^{n-1}}{5^n} = \sum_{n=0}^{+\infty}\frac{2^n\cdot 2^{-1}}{5^n} = \frac{1}{2}\sum_{n=0}^{+\infty}\left(\frac{2}{5}\right)^n =$$

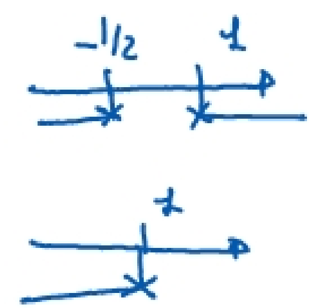
$$= \frac{1}{2}\cdot\frac{1}{1-\frac{2}{5}} = \frac{1}{2}\cdot\frac{5}{3} = \frac{5}{6}$$

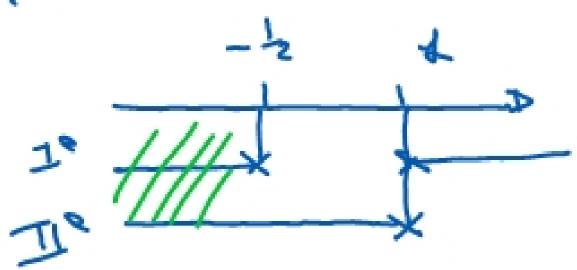
es 8 Dire per quali valori del parametro $a \in \mathbb{R}$, la serie $\sum_{n=0}^{+\infty}\left(\frac{2+a}{1-a}\right)^n$ converge, in caso affermativo calcolare s.

① serie geometrica $q=\frac{2+a}{1-a}$, convergenza se

$$|q|<1 \iff \left|\frac{2+a}{1-a}\right|<1 \Rightarrow -1<\frac{2+a}{1-a}<1$$

$$\begin{cases} \frac{2+a}{1-a}<1 \\ \frac{2+a}{1-a}>-1 \end{cases} \qquad \begin{cases} \frac{2+a-1+a}{1-a}<0 \\ \frac{2+a+1-a}{1-a}>0 \end{cases} \qquad \begin{cases} \frac{2a+1}{1-a}<0 \\ \frac{3}{1-a}>0 \end{cases}$$





$a<-\frac{1}{2}$ convergente

② $$\sum_{n=0}^{+\infty}\left(\frac{2+a}{1-a}\right)^n=\frac{1}{1-\frac{2+a}{1-a}}=\frac{1-a}{1-a-2-a}=\frac{1-a}{-1-2a}=\frac{a-1}{2a+1}$$

con $a<-\frac{1}{2}$