EQUAZIONI LINEARI DEL PRIMO ORDINE
A COEFFICIENTI CONTINUI

$y'(x) + a(x)\, y(x) = b(x)$ oppure $\begin{cases} y'(x) + a(x)\, y(x) = b(x) \\ y(x_0) = y_0 \end{cases}$

$\Downarrow$ $\Downarrow$

$$y(x) = e^{-A(x)} \left[y_0 + \int_{x_0}^{x} e^{A(s)}\, b(s)\, ds \right]$$

$y_0 = c \quad c \in \mathbb{R}$ $\Leftarrow$ $A(x) = \int_{x_0}^{x} a(s)\, ds$

$\Downarrow$

$$y(x) = e^{-A(x)} \left[c + \int e^{A(x)}\, b(x)\, dx \right.$$

es 1 $\quad y' - xy = x^3$

$a(x) = -x \qquad A(x) = \int a(x)\,dx = \int -x\,dx = -\frac{x^2}{2} + c_1$

$b(x) = x^3$

$$\int e^{A(x)}\, b(x)\,dx = \int e^{-\frac{x^2}{2}} x^3\,dx$$

[N.B. $\quad \int e^{-\frac{x^2}{2}+c_1} x^3\,dx = \int e^{-\frac{x^2}{2}} \cdot e^{c_1} \cdot x^3\,dx =$

$$= \overset{e^{c_1}}{\overset{\parallel}{c_2}} \int e^{-\frac{x^2}{2}} x^3\,dx$$

$$\int e^{-\frac{x^2}{2}} x^3\,dx = -\int \underbrace{-e^{-\frac{x^2}{2}} \cdot x}_{f'} \cdot \underbrace{x^2}_{g}\,dx \underset{\text{p.p.}}{=}$$

$D\left(-\frac{x^2}{2}\right) = -x$

$$= -\int \underbrace{-e^{-\frac{x^2}{2}} \cdot x}_{f'} \cdot \underbrace{x^2}_{g} \, dx = -e^{-\frac{x^2}{2}} \cdot x^2 + \underbrace{\int e^{-\frac{x^2}{2}} \cdot 2x \, dx}_{} =$$

$$= -e^{-\frac{x^2}{2}} \cdot x^2 - 2 \int \underbrace{-e^{-\frac{x^2}{2}} \cdot x}_{D\, e^{-\frac{x^2}{2}}} \, dx = \underline{-e^{-\frac{x^2}{2}} x^2 - 2e^{-\frac{x^2}{2}} + c_2}$$

$$= -e^{-\frac{x^2}{2}}(x^2+2)$$

Soluzione:

$$y(x) = c\, e^{x^2/2} + e^{\frac{x^2}{2}} \left(-e^{-\frac{x^2}{2}}(x^2+2) + c_2 \right) =$$

$$= \underbrace{c e^{x^2/2}}_{} - (x^2+2) \underbrace{+ c_2 e^{\frac{x^2}{2}}}_{} =$$

$$= (c + c_2)\, e^{\frac{x^2}{2}} - (x^2+2) =$$

$$= K e^{x^2/2} - (x^2+2)$$

Problema di Cauchy:

$$\begin{cases} y' - xy = x^3 \\ y(1) = 3 \end{cases}$$

problema in generale: $y = k\, e^{\frac{x^2}{2}} - (x^2+2)$

$$y(1) = 3 \quad \longrightarrow \quad 3 = k\, e^{\frac{1^2}{2}} - (1^2+2)$$

$$3 = k\, e^{1/2} - 3$$

$$k\, e^{1/2} = 6$$

$$k = e^{-1/2} \cdot 6$$

$$\Rightarrow \quad y = 6e^{-\frac{1}{2}}\, e^{\frac{x^2}{2}} - (x^2+2)$$

ES 2

$$y' + y\,\overbrace{\cos x}^{a(x)} = \overbrace{\operatorname{sen} x \cos x}^{b(x)}$$

$$y = e^{-\int \cos x \, dx}\left[\int \operatorname{sen} x \cos x \cdot e^{\int \cos x} dx + c\right] = (*)$$

$$A(x) = -\int \cos x \, dx = -\operatorname{sen} x + c_1$$

$$\int \underbrace{\operatorname{sen} x}_{g(x)} \underbrace{\cos x \; e^{\operatorname{sen} x}}_{f'(x)} dx \underset{p.p.}{=} e^{\operatorname{sen} x} \operatorname{sen} x - \int e^{\operatorname{sen} x} \cdot \cos x \, dx =$$

$$= e^{\operatorname{sen} x} \operatorname{sen} x - e^{\operatorname{sen} x} + c_2$$

$$(*) = e^{-\operatorname{sen} x}\left[e^{\operatorname{sen} x} \operatorname{sen} x - e^{\operatorname{sen} x} + c\right]$$

es 3 $\quad y' = -\frac{2}{x} y + \frac{\operatorname{sen} 4x}{x^2}$

$$y' + \underbrace{\frac{2}{x}}_{a(x)} y = \underbrace{\frac{\operatorname{sen} 4x}{x^2}}_{b(x)}$$

$$y = e^{-\int \frac{2}{x} dx} \left[\int \frac{\operatorname{sen} 4x}{x^2} \cdot e^{\int \frac{2}{x} dx} dx + c \right]$$

1. $\int \frac{2}{x} dx = 2 \ln |x| + c_1 = \ln x^2 + c_1$

$$e^{-\int \frac{2}{x} dx} = e^{-\ln x^2} = e^{\ln (x^{-2})} = x^{-2} = \frac{1}{x^2}$$

$$e^{\int \frac{2}{x} dx} = e^{\ln x^2} = x^2$$

2. $\int \frac{\operatorname{sen} 4x}{x^2} \cdot e^{\int \frac{2}{x} dx} dx = \int \frac{\operatorname{sen} 4x}{\cancel{x^2}} \cdot \cancel{x^2} dx = \int \operatorname{sen} 4x \, dx =$

$$= \int \operatorname{sen} 4x \, dx = \frac{-\cos 4x}{4} + C_2$$

$$(*) \quad y = \frac{1}{x^2} \left[-\frac{\cos 4x}{4} + C \right]$$

es 4 $\quad y' = y + x^2$

$$y' - y = x^2 \qquad a(x) = -1$$

$$b(x) = x^2$$

$$y = e^{-\int - dx}\left[\int x^2 e^{\int - dx}\,dx + c\right] = e^{x}\left[-x^2 e^{-x} - 2x e^{-x} - 2e^{-x} + c\right]$$

$$= e^{x}\left[-e^{-x}(x^2+2x+2) + c\right]$$

(*)

$$-\int -x^2 e^{-x}\,dx \underset{\text{p.p}}{=} -x^2 e^{-x} + \int 2x e^{-x}\,dx = -x^2 e^{-x} - 2\int -x e^{-x}\,dx =$$

$$D(e^{-x}) = -e^{-x}$$

$$\underset{\text{p.p}}{=} -x^2 e^{-x} - 2\left(x e^{-x} - \int e^{-x}\,dx\right) = \underbrace{-x^2 e^{-x} - 2\left(x e^{-x} + e^{-x}\right)}_{(*)} + c_2$$

es 5

$$y' - \frac{1}{x} y = - \frac{3x+2}{x^2}$$

$$A(x) = \int -\frac{1}{x}\, dx = -\ln|x| + c_1$$

$$B(x) = \int e^{A(x)} \cdot b(x)\, dx = \int e^{-\ln|x|} \cdot \left(- \frac{3x+2}{x^2} \right) dx =$$

$$= \int e^{\ln|x|^{-1}} \left(- \frac{3x+2}{x^2} \right) dx = \int - \frac{1}{|x|} \frac{3x+2}{x^2}\, dx - \int \frac{3x+2}{|x|^3}\, dx =$$

$$= - \int \left[\frac{-3}{-x^2} + \frac{2(-4)}{(-4)|x|^3} \right] dx = - \left[-\frac{1}{x^3} - \frac{1}{2} x^{-4} \right] + c_2$$

$$\Rightarrow \quad y = e^{-(-\ln|x|)} \left[c + \frac{1}{x^3} + \frac{1}{2} \frac{1}{x^4} \right] = |x| \left[c + \frac{1}{x^3} + \frac{1}{2} \frac{1}{x^4} \right]$$

es 6

$$\begin{cases} y' - \frac{1}{\cos^2 x}\, y = 12\, e^{\operatorname{Tg} x} \cos 2x \\ y(0) = 0 \end{cases}$$

$$A(x) = \int_0^x -\frac{1}{\cos^2 s}\, ds = -\operatorname{Tg} s \Big|_0^x = -\operatorname{Tg} x$$

$$B(x) = \int_0^x e^{-\operatorname{Tg} s} \cdot 12\, e^{\operatorname{Tg} s} \cos 2s\, ds = 12 \int_0^x \cos 2s\, ds =$$

$$= 12 \cdot \frac{1}{2} \operatorname{sen} 2s \Big|_0^x = 6 \operatorname{sen} 2x$$

Soluzione $\quad y = e^{-(-\operatorname{Tg} x)} \left[0 + 6 \operatorname{sen} 2x \right] = e^{\operatorname{Tg} x} \operatorname{sen} 2$

es 7

$$\begin{cases} y' + \frac{1}{x} y = -9 \, \frac{\cos 3x}{x} \\ y(\pi) = 0 \end{cases}$$

$$A(x) = \int_\pi^x \frac{1}{s} \, ds = \ln|s| \Big|_\pi^x = \ln|x| - \ln \pi$$

$$B(x) = \int_\pi^x e^{\ln|s| - \ln\pi} \left(-9 \, \frac{\cos 3s}{s}\right) ds = \int_\pi^x e^{\ln|s|} \cdot e^{\ln(\pi^{-1})} \left(\frac{-9 \cos 3s}{s}\right) ds =$$

$$= -9 \cdot \pi^{-1} \int_\pi^x \cancel{s} \cdot \frac{\cos 3s}{\cancel{s}} \, ds = -\frac{9}{\pi} \left[\frac{\operatorname{sen} 3s}{3} \right]_\pi^x = -\frac{\cancel{9}^{\,3}}{\pi} \, \frac{\operatorname{sen} 3x}{\cancel{3}_{\,1}} =$$

$((\pi \to x) > 0)$

$$= -\frac{3}{\pi} \operatorname{sen} 3x \qquad [\text{N.B. } \operatorname{sen} 3\pi = 0]$$

soluzione:

$$y = e^{-[\ln|x| - \ln\pi]} \left[0 - \frac{3}{\pi} \operatorname{sen} 3x \right] =$$

$$= e^{\ln|x|^{-1}} \cdot \frac{1}{\pi} \cdot \left(-\frac{3}{\pi} \operatorname{sen} 3x \right) = -\frac{1}{|x|} \cdot \frac{3}{\pi^2} \operatorname{sen} 3x$$

es 8 $\begin{cases} y' - \frac{1}{x} y = \frac{2e^{-1/x^2}}{x^2} - \frac{1}{xe} \\ y(1) = \frac{2}{e} \end{cases}$

$$A(x) = \int_1^x -\frac{1}{s}\, ds = -\ln|s| \Big|_1^x = \ln\left(\frac{1}{|x|}\right) - \underbrace{\ln 1}_{0}$$

$$B(x) = \int_1^x e^{\ln\frac{1}{|s|}} \cdot \left[\frac{2e^{-\frac{1}{s^2}}}{s^2} - \frac{1}{se}\right] ds = \int_1^x \left(\frac{1}{|s|} \cdot \frac{2e^{-\frac{1}{s^2}}}{s^2} - \frac{1}{se} \cdot \frac{1}{|s|}\right) ds =$$

$(1 \to x) > 0$

$$= \int_1^x + 2\frac{e^{-\frac{1}{s^2}}}{s^3}\, ds - \int_1^x \frac{1}{es^2}\, ds = \left[e^{-\frac{1}{s^2}} + \frac{1}{es}\right]_1^x =$$

$$+\frac{2}{s^3} = D\left(-\frac{1}{s^2}\right)$$

$$= e^{-1/x^2} + \frac{1}{ex} - e^{-1} - \frac{1}{e} =$$

la soluzione è

$$y = e^{-(-\ln|x|)}\left[\cancel{\frac{2}{e}} + e^{-\frac{1}{x^2}} + \frac{1}{ex} - \cancel{\frac{2}{e}}\right] = |x|\left(e^{-\frac{1}{x^2}} + \frac{1}{ex}\right)$$