SUCCESSIONI 2

$$\lim_{n\to+\infty} \frac{(2e)^n \sqrt{n^2+1} \quad n!}{(2n)^n \; n^2} =$$

(esercizio tratto dalle dispense)

$$= \lim_{n\to+\infty} \frac{(2e)^n \sqrt{n^2+1} \cdot n^n e^{-n} \sqrt{2\pi n}\; e^{\frac{\vartheta_n}{12n}}}{(2n)^n \; n^2} =$$

FORMULA DI DEMOIVRE-
- STIRLING

$$n! = n^n e^{-n} \sqrt{2\pi n}\, e^{\frac{\vartheta_n}{12n}}$$

con $0 < \vartheta_n < 1$

$$= \lim_{n\to+\infty} \frac{\cancel{2^n}\, \cancel{e^n} \sqrt{n^2+1} \;\; \cancel{n^n}\, \cancel{e^{-n}} \sqrt{2\pi n} \;\; e^{\frac{\vartheta_n}{12n}}}{\cancel{2^n}\; \cancel{n^n}\; n^2} =$$

$$= \lim_{n\to+\infty} \frac{\sqrt{n^2+1} \; \sqrt{2\pi n} \;\; e^{\overbrace{\frac{\vartheta_n}{12n}}^{\to 0}}}{n^2} =$$

$$\simeq \lim_{n\to+\infty} \frac{\cancel{n} \sqrt{1 + \overbrace{\frac{1}{n^2}}^{\to 0}} \;\; \sqrt{n} \cdot \sqrt{2\pi}}{n^{\cancel{2}}} = 0$$

$$\lim_{n\to+\infty} \frac{n - n^{\frac{1}{n}+1}}{\log n^5} =$$

$$= \lim_{n\to+\infty} \frac{n\left(1 - n^{\frac{1}{n}}\right)}{5 \log n} =$$

$$\log n^{\frac{1}{n}}$$

$$= \lim_{n\to+\infty} \frac{1 - e^{\frac{\log n}{n}}}{5\, \frac{\log n}{n}} =$$

$$= \lim_{n\to+\infty} \left(-\frac{1}{5}\right) \frac{e^{\frac{\log n}{n}} - 1}{\frac{\log n}{n}} = -\frac{1}{5} \cdot 1 = -\frac{1}{5}$$

F.I.

$$\lim_{t\to 0} \frac{e^{t} - 1}{t} = 1$$

$\Leftarrow$ attenzione $b_n = \frac{\log n}{n}$: $\lim_{n\to+\infty} b_n = 0$!

$$\boxed{n^{\frac{1}{n}} \rightsquigarrow \infty^0}$$

$\Downarrow$

passare all'esponenziale

$$n^{\frac{1}{n}} = e^{\log n^{\frac{1}{n}}}$$

$$\lim_{n\to+\infty} \frac{\log n}{n} = 0$$

$$\lim_{n\to+\infty} \frac{4^n - 2^{n(n-1)!}}{n^n} =$$

$$= \lim_{n\to+\infty} \left[\left(\frac{4}{n}\right)^n - \left(\frac{2^{(n-1)!}}{n}\right)^n \right] = (*)$$

$$0 < \frac{4}{n} < 1 \qquad n > 4 \qquad \lim_{n\to+\infty} \left(\frac{4}{n}\right)^n \leq \lim_{n\to+\infty} \left(\frac{4}{5}\right)^n = 0$$

$$b_n = \frac{2^{(n-1)!}}{n}$$

$$b_2 = \frac{2^{1!}}{2} = \frac{2}{2} = 1$$

$$b_3 = \frac{2^{2!}}{3} = \frac{4}{3} = \frac{2^2}{3} > 1$$

$$b_4 = \frac{2^{3!}}{4} = \frac{2^6}{2^2} = 2^4$$

$$b_n = \frac{2^{(n-1)!}}{n} > 2$$

$$(*) = -\lim_{n\to+\infty} \left(\frac{2^{(n-1)!}}{n}\right)^n \leq$$

$$\leq -\lim_{n\to+\infty} 2^n = -\infty$$

$$\lim_{n\to+\infty} \frac{(n+1)^n + e^n \log n}{(n-1)^n - n!\cos n\frac{\pi}{2}} =$$

$$= \lim_{n\to+\infty} \frac{\cancel{n^n}\left(\frac{(n+1)^n}{n^n} + \frac{e^n \log n}{n^n}\right)}{\cancel{n^n}\left(\frac{(n-1)^n}{n^n} - \frac{n!}{n^n}\cos n\frac{\pi}{2}\right)} =$$

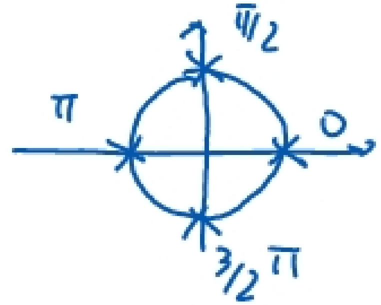
$$= \lim_{n\to+\infty} \frac{\underbrace{\left(\frac{n+1}{n}\right)^n}_{\to e} + \frac{e^n \log n}{n^n}}{\underbrace{\left(\frac{n-1}{n}\right)^n}_{\to e^{-1}} - \underbrace{\frac{n!}{n^n}}_{\to 0}\cos n\frac{\pi}{2}} = \frac{e}{e^{-1}} = e^2$$

$$\lim_{n\to+\infty} \frac{e^n \log n}{n^n} =$$

$$= \lim_{n\to+\infty} \frac{\overbrace{e^{n-1}\cdot e}^{e^n}}{\underbrace{n^{n-1}\cdot n}_{n^n}} \cdot \log n =$$

$$= \lim_{n\to+\infty} \underbrace{\frac{e^{n-1}}{n^{n-1}}}_{\to 0} \cdot e \cdot \underbrace{\frac{\log n}{n}}_{\to 0} = 0$$

$$b_n = \cos\left(n\frac{\pi}{2}\right) = \begin{cases} 0 & n \text{ dispari} \\ +1 \\ -1 & n \text{ pari} \end{cases}$$



• determinare $\alpha \in \mathbb{R}$: $\lim\limits_{n\to+\infty} \dfrac{(n+2)\left(3\,\operatorname{arctg}\sqrt{n} + n^{2\alpha-1}\right)}{n+7}$ vale ...

(Tema d'esame 4-07-05)

$$= \lim_{n\to+\infty} \underbrace{\left(\frac{n+2}{n+7}\right)}_{\to 1} n^{2\alpha-1}\left(3\,\frac{\operatorname{arctg}\sqrt{n}}{n^{2\alpha-1}} + 1\right) \simeq$$

$$\simeq \lim_{n\to+\infty} n^{2\alpha-1}\left(3\,\frac{\operatorname{arctg}\sqrt{n}}{n^{2\alpha-1}} + 1\right) \qquad \operatorname{arctg}\sqrt{n} \underset{n\to+\infty}{\longrightarrow} \pi/2$$

se $\left(2\alpha-1>0 \quad \alpha>\frac{1}{2}\right)$ $\quad n^{2\alpha-1}\to+\infty \quad \dfrac{3\,\operatorname{arctg}\sqrt{n}}{n^{2\alpha-1}}\to 0 \quad \lim\limits_{n\to+\infty}(\ ) = +\infty$

se $\left(2\alpha-1=0 \quad \alpha=\frac{1}{2}\right)$ $\quad n^{2\alpha-1}=n^0=1 \quad \dfrac{3\,\operatorname{arctg}\sqrt{n}}{1}\to 3\frac{\pi}{2} \quad \lim\limits_{n\to+\infty}(\ ) = \frac{3}{2}\pi+$

se $\left(2\alpha-1<0 \quad \alpha<\frac{1}{2}\right)$

$$= \lim_{n\to+\infty}\left(3\,\operatorname{arctg}\sqrt{n} + \underbrace{n^{2\alpha-1}}_{\to 0}\right) = \lim_{n\to+\infty} 3\,\operatorname{arctg}\sqrt{n} = \frac{3}{2}\pi$$

$$\lim_{n\to+\infty} n^2\left(\log(n+2)+\log\frac{1}{n}\right)\sin\frac{1}{n} =$$

(Tema d'esame del 8-12-04)

$$= \lim_{n\to+\infty} n\left(\log\frac{n+2}{n}\right)\underbrace{\frac{\sin\frac{1}{n}}{\frac{1}{n}}}_{\to 1} =$$

(limite notevole)

$$= \lim_{n\to+\infty} \log\left[\underbrace{\left(1+\frac{2}{n}\right)^{\frac{n}{2}}}_{e}\right]^{2} = \log e^{2} = 2\,\underbrace{\log e}_{1} = 2$$

(limite notevole)

$$\lim_{n\to+\infty}\left(\frac{\log\left(1+\frac{1}{2n}\right)}{7\log n\cdot\sin\frac{1}{2n}}+\frac{\log(n+2)+n^{-\frac{1}{2}}}{2\log n}\right)=$$

$$\overbrace{\log n+\underbrace{\log\left(1+\tfrac{2}{n}\right)}_{\to 0}}$$

$$=\lim_{n\to+\infty}\left(\frac{\overbrace{\log\left(1+\frac{1}{2n}\right)}^{\to 1}}{7\log n\cdot\underbrace{\frac{\sin\frac{1}{2n}}{\frac{1}{2n}}}_{1}\cdot\frac{1}{2n}}+\underbrace{\frac{\log\left(n\left(1+\frac{2}{n}\right)\right)+\overbrace{\frac{1}{\sqrt{n}}}^{\to 0}}{2\log n}}_{\sim\ \frac{\cancel{\log n}}{2\,\cancel{\log n}}}\right)=$$

$$\frac{1}{\infty}=0$$

$$=\quad 0\quad+\frac{1}{2}=\frac{1}{2}$$

(Tema d'esame del 26-01-09)

$$\lim_{n\to+\infty} \frac{n^n + 7n\log n + n\sin n}{(n+2)^n + n\log\frac{1}{n} + n\sin\frac{1}{n}} =$$

$$= \lim_{n\to+\infty} \frac{n^n\left(1 + \frac{7n\log n}{n^n} + \frac{n\sin n}{n^n}\right)}{(n+2)^n\left(1 + \frac{n\log\frac{1}{n}}{(n+2)^n} + \frac{n\sin\frac{1}{n}}{(n+2)^n}\right)} =$$

$$= \lim_{n\to+\infty} \frac{n^n}{(n+2)^n} = \lim_{n\to+\infty}\left(\frac{n}{n+2}\right)^n = \lim_{n\to+\infty}\left(\frac{n+2}{n}\right)^{-n} =$$

$$= \lim_{n\to+\infty}\left[\left(1+\frac{2}{n}\right)^{\frac{n}{2}}\right]^{-2} = e^{-2}$$

NUM

1) $\dfrac{7\, n \log n}{n^n} = 7\, \dfrac{n^2}{n^n}^{\to 0} \cdot \dfrac{\log n}{n}^{\to 0} \to 0$

2) $\dfrac{n \sin n}{n^n} = \dfrac{n}{n^n}^{\to 0} \sin n^{[-1,1]} \to 0$

DEN

3) $\dfrac{n \log \frac{1}{n}}{(n+2)^n} = \dfrac{n \log n^{-1}}{(n+2)^n} = \dfrac{-n \log n}{(n+2)^n} = -\dfrac{n^2}{(n+2)^n}^{\to 0} \cdot \dfrac{\log n}{n}^{\to 0} \to 0$

4) $\dfrac{n \sin \frac{1}{n}}{(n+2)^n} \to \dfrac{n \sin \frac{1}{n}^{\to 1}}{\underbrace{(n+2)^n}_{\to +\infty}} \to 0$ (limite notevole)

RIEPILOGO SUL CALCOLO DEI LIMITI

RICORDARSI CHE:

$$\lim_{n\to+\infty} \sqrt[n]{n} = 1$$

$$\lim_{n\to+\infty} \sqrt[n]{\frac{n!}{n^n}} = \frac{1}{e}$$

$$\lim_{n\to+\infty} \frac{n!}{n^n} = 0$$

$$\lim_{n\to+\infty} \sqrt[n]{n!} = +\infty$$

$$\lim_{n\to+\infty} n\left(e^{\frac{1}{n}}-1\right) = 1$$

FORMA DI INDETERMINAZIONE:

$+\infty - \infty$

1. "razionalizzazione inversa"
2. proprietà operatori di calcolo
3. raccoglimento del "più forte"

$\frac{\infty}{\infty}$ o $\frac{0}{0}$

1. confronto i gradi dei polinomi N, D
2. raccoglimento
3. scomposizione
4. raccoglimento forzato
5. confronto fra infiniti, come ordine

ORDINE DI INFINITO (CATENA DEL CONFRONTO):

$$\log_a n < n^\alpha < a^n < n! < n^n$$

con $a>1$ / $\alpha \in \mathbb{N}$

LIMITI NOTEVOLI

(1)

$$\lim_{n \to +\infty} \frac{\operatorname{sen}(a_n)}{a_n} = 1$$

con $a_n \rightsquigarrow 0$

quindi

$$\lim_{n \to +\infty} \frac{\operatorname{sen}\left(\frac{1}{n}\right)}{1/n} = 1$$

$$\lim_{n \to +\infty} n \operatorname{sen}\left(\frac{1}{n}\right) = 1$$

TEOREMI: 1. confronto fra successioni
2. teorema dei Carabinieri
3. criterio del rapporto

(2)

$$\lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^n = e$$

quindi: $\lim_{n \to +\infty} n \log\left(1 + \frac{1}{n}\right) = 1$

$$\lim_{n \to +\infty} \left(1 + \frac{1}{a_n}\right)^{a_n} = e$$

se $a_n \rightsquigarrow +\infty$

$$\lim_{n \to +\infty} \left(1 + \frac{1}{a_n}\right)^{b_n} = e^{\lim \frac{b_n}{a_n}}$$

se $a_n \rightsquigarrow +\infty$
$b_n \rightsquigarrow +\infty$

FORMA DI INDETERMINAZIONE

$(a_n)^{b_n}$: 1^∞, ∞^0, 0^0 $\Rightarrow$ si passa alla forma esponenziale $e^{\log(a_n)^{b_n}} = e^{b_n \cdot \log(a_n)}$