NUMERI COMPLESSI (2ª parte)

(Tema d'esame 10-08.07)

es 1 Determinare il luogo geometrico degli $z \in \mathbb{C}$ tali che:

$$\operatorname{Im}\left(\frac{|z|-2i}{|z|+2i}\right)+1=7\operatorname{Re}(z-\bar{z}) \quad (*)$$

$z = x+iy$

$|z| = \sqrt{x^2+y^2}$

$$\frac{|z|-2i}{|z|+2i}=\frac{\sqrt{x^2+y^2}-2i}{\sqrt{x^2+y^2}+2i}\cdot\frac{\sqrt{x^2+y^2}-2i}{\sqrt{x^2+y^2}-2i}=$$

$$=\frac{\left(\sqrt{x^2+y^2}-2i\right)^2}{x^2+y^2+4}=\frac{x^2+y^2-4-4i\sqrt{x^2+y^2}}{x^2+y^2+4}=$$

$$=\underbrace{\frac{x^2+y^2-4}{x^2+y^2+4}}_{\operatorname{Re}}-\underbrace{\frac{4\sqrt{x^2+y^2}}{x^2+y^2+4}}_{\operatorname{Im}}\,i$$

denominatori
$x^2+y^2+4 \neq 0$
$\forall (x,y) \in \mathbb{R}$

$$\operatorname{Im}\left(\frac{|z|-2i}{|z|+2i}\right)=-\frac{4\sqrt{x^2+y^2}}{x^2+y^2+4}$$

$7\ \mathrm{Re}\,(z-\bar{z}) = 0$

$$z-\bar{z} = x+iy-(x-iy) = x+iy-x+iy = 2yi$$

$$(*) \quad \frac{-4\sqrt{x^2+y^2}}{x^2+y^2+4} + 1 = 0$$

$$\left[4\sqrt{x^2+y^2}\right]^2 = \left[(x^2+y^2)+4\right]^2$$

$$16(x^2+y^2) = (x^2+y^2) + 16 + 8(x^2+y^2)$$

$$(x^2+y^2)^2 - 8(x^2+y^2) + 16 = 0$$

$$(x^2+y^2-4)^2 = 0 \iff x^2+y^2-4 = 0 \quad \text{cfr}$$

$$x^2+y^2 = 4 \qquad C(0,0) \quad R = 2$$

Il luogo cercato è una circonferenza centrata nell'origine e raggio 2

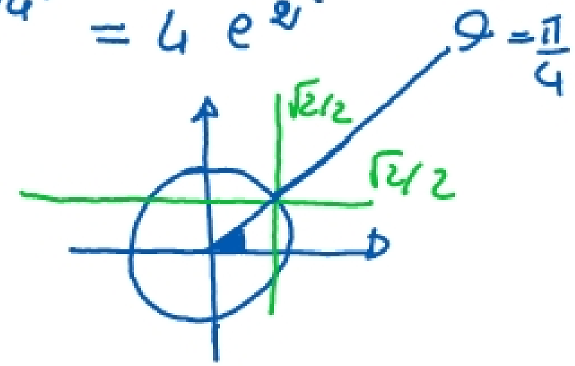
(Tema d'esame 10_12_07)

es 2: Si determini il numero complesso:

$$w = 2(\sqrt{2} + \sqrt{2}\, i)\, e^{i\frac{\pi}{4}}$$

e se ne scrivano le sue radici cubiche in forma algebrica/cartesiana

$$w = 2\sqrt{2}\,(1+i)\, e^{i\frac{\pi}{4}} = 2\sqrt{2}\cdot \underbrace{\sqrt{2}\cdot e^{\frac{\pi}{4}i}}_{z} \cdot e^{\pi/4\, i} = 4\, e^{\frac{\pi}{2}i}$$



$z = 1 + i$

$\rho = \sqrt{1+1} = \sqrt{2}$

$$\vartheta = \begin{cases} \cos\vartheta = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \\ \operatorname{sen}\vartheta = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \end{cases} \qquad \vartheta = \pi/4$$

$w = 4\, e^{\frac{\pi}{2}i} \qquad \rho = 4 \qquad \vartheta = \frac{\pi}{2}$

$\vartheta_0 = \left(\frac{\pi}{2} + 2\cdot 0\cdot \pi\right)/3 = \frac{\pi}{6} \qquad w_0 = \sqrt[3]{4}\cdot\left(\cos\frac{\pi}{6} + i \operatorname{sen}\frac{\pi}{6}\right) = \sqrt[3]{4}\left(\frac{\sqrt{3}}{2} + \frac{1}{2}\, i\right)$

$\vartheta_1 = \left(\frac{\pi}{2} + 2\cdot 1\cdot \pi\right)/3 = \frac{5}{6}\pi \qquad w_1 = \sqrt[3]{4}\left(\cos\frac{5}{6}\pi + i \operatorname{sen}\frac{5}{6}\pi\right) = \sqrt[3]{4}\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}\, i\right)$

$\vartheta_2 = \left(\frac{\pi}{2} + 2\cdot 2\cdot \pi\right)/3 = \frac{9}{6}\pi = \frac{3}{2}\pi \qquad w_2 = \sqrt[3]{4}\left(\underbrace{\cos\frac{3}{2}\pi}_{=0} + i \operatorname{sen}\frac{3}{2}\pi\right) = \sqrt[3]{4}\,(-i) = -\sqrt[3]{4}\, i$

(Tema d'esame 10.12.07)

es 3 Determinare il luogo geometrico degli $z \in \mathbb{C}$ tali che:

$$\left[Re\left(i\bar{z}(z-2)\right)\right]^2 - \left[Im\left(z(\bar{z}-2i)\right)\right]^2 = 0 \qquad z = x+iy$$

$$i\bar{z}(z-2) = i(x-iy)(x+iy-2) = i(x-iy)\left((x-2)+iy\right) =$$

$$= i\left\{\left[x(x-2)+y^2\right] + i\left(xy - y(x-2)\right)\right\} =$$

$$= \underbrace{-\left(xy - y(x-2)\right)}_{Re} + i\underbrace{\left(x(x-2)+y^2\right)}_{Im}$$

$$z(\bar{z}-2i) = (x+iy)(x-iy-2i) = (x+iy)\left(x-(y+2)i\right) =$$

$$= \underbrace{x^2+y(y+2)}_{Re} + i\underbrace{\left(xy - x(y+2)\right)}_{Im}$$

$$\left[-(xy - y(x-2))\right]^2 - \left[xy - x(y+2)\right]^2 = 0$$

$$\left[-(xy - xy + 2y)\right]^2 - \left[xy - xy - 2x\right]^2 = 0$$

$$4y^2 - 4x^2 = 0$$

$$4(y^2 - x^2) = 0$$

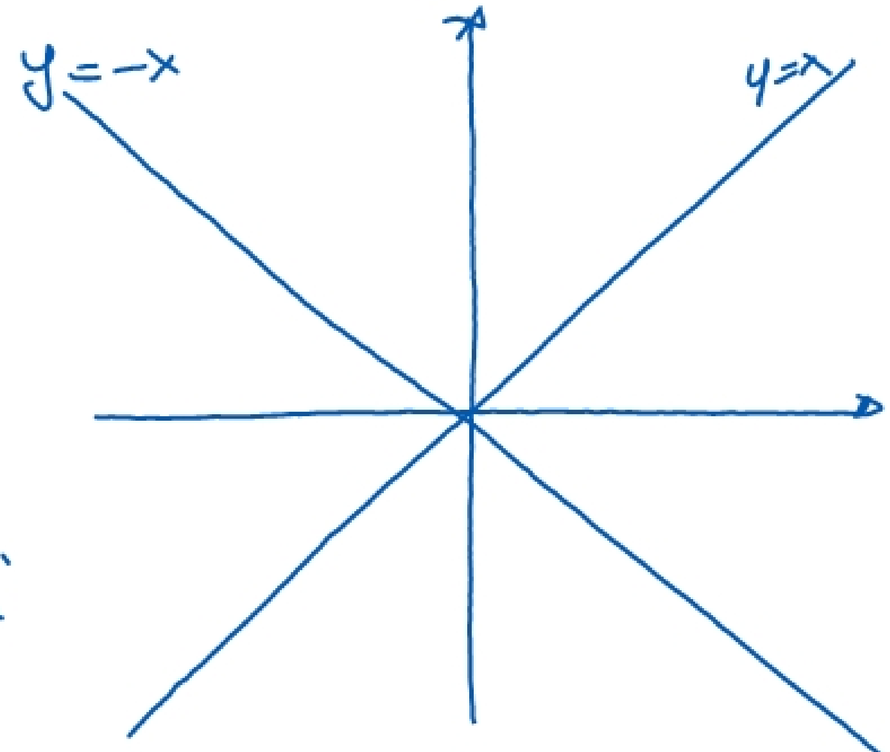
$$4(y-x)(y+x) = 0$$

se $y - x = 0 \to y = x$

bisettrici

oppure

$y + x = 0 \to y = -x$



Il luogo geometrico sono le bisettrici, (2 rette),

(Tema d'esame 18-03-08)

es 4: Determinare il luogo geometrico degli $z \in \mathbb{C}$ tali che:

$$2\,|z+7i| = |\bar{z} - 14i| \qquad z = x+iy$$

$z+7i = x+iy+7i = x+(y+7)\,i \qquad |z+7i| = \sqrt{x^2+(y+7)^2}$

$\bar{z}-14i = x-iy-14i = x-(y+14)\,i \qquad |\bar{z}-14i| = \sqrt{x^2+(y+14)^2}$

$$2\sqrt{x^2+(y+7)^2} = \sqrt{x^2+(y+14)^2} \qquad \text{(cond. radicandi)}$$

$$4\left(x^2+(y+7)^2\right) = \left(x^2+(y+14)^2\right)$$

$$4x^2+\underbrace{4\,(y^2+49}_{196}+14y) = x^2+y^2+28y+\cancel{196}$$

$$3x^2+3y^2+28y = 0 \;\rightarrow\; x^2+y^2+\frac{28}{3}y = 0 \qquad \text{cfr ?}$$

$\alpha = 0$

$\beta = -\frac{28}{6} = -\frac{14}{3} \qquad R = \sqrt{\underset{0}{\alpha^2}+\beta^2-\underset{0}{c}} = \frac{14}{3}$

Il luogo è una circonferenza $\left(C\left(0,-\frac{14}{3}\right) \quad R = \frac{14}{3}\right)$

(Tema d'esame 18-03-08)

es5: Calcolare in $\mathbb{C}$ tutte le soluzioni, contate con la loro molteplicità, della seguente equazione:

$$[z^2+4iz-4]\,(z^3-i)=0$$ 5° grado → 5 soluzioni in $\mathbb{C}$

1) $z^2+4iz-4=0$ $\quad (z+2i)^2=0$ $\quad z=-2i$ molteplicità 2

2) $z^3-i=0$ $\quad z^3=i=e^{\frac{\pi}{2}i}$

$\rho=1 \quad \vartheta=\pi/2 \quad n=3$

$\vartheta_k=\dfrac{\theta+2k\pi}{n}$ $\quad \vartheta_0=\left(\frac{\pi}{2}+0\pi\right)/3=\pi/6$ $\quad z_0=e^{\pi/6\, i}$

$\rho_k=\sqrt[3]{1}=1$ $\quad \vartheta_1=\left(\frac{\pi}{2}+2\pi\right)/3=\frac{5}{6}\pi$ $\quad z_1=e^{5/6\,\pi i}$

$\vartheta_2=\left(\frac{\pi}{2}+4\pi\right)/3=3/2\,\pi$ $\quad z_2=e^{3/2\,\pi i}$

z_0, z_1, z_2: molteplicità 1

es 6: Determinare il luogo geometrico degli $z \in \mathbb{C}$ tali che:

(Tema d'esame 03/07/08)

$$|z-7|^2 + |z+7|^2 = 2(z-7)^2$$

$$x + iy = z$$

$$z-7 = x+iy-7 = (x-7)+iy \qquad |z-7|^2 = (x-7)^2+y^2$$

$$z+7 = x+iy+7 = (x+7)+iy \qquad |z+7|^2 = (x+7)^2+y^2$$

$$(z-7)^2 = (x+iy-7)^2 = \big((x-7)+iy\big)^2 = \underbrace{(x-7)^2 - y^2 + 2y(x-7)\,i}$$

$$\left[(x-7)^2+y^2\right] + \left[(x+7)^2+y^2\right] = 2\left[(x-7)^2 - y^2 + 2y(x-7)\,i\right]$$

$$\underbrace{(x-7)^2+(x+7)^2+2y^2}_{Re} \underset{Im=0}{=} \underbrace{2\left[(x-7)^2-y^2\right]}_{Re} + \underbrace{4y(x-7)\,i}_{Im}$$

$$\begin{cases} (x-7)^2+(x+7)^2+2y^2 = 2(x-7)^2-2y^2 \\ 4y(x-7) = 0 \end{cases}$$

$$\begin{cases} (x-7)^2-(x+7)^2-4y^2=0 \\ y\,(x-7)=0 \end{cases}$$

$$\begin{cases} y=0 \\ (x-7)^2-(x+7)^2=0 \end{cases}$$

$$\begin{cases} y=0 \\ (x-7+x+7)\,(x-7-x-7)=0 \end{cases}$$

$$\begin{cases} y=0 \\ -14\,(2x)=0 \end{cases}$$

$$\begin{cases} y=0 \\ x=0 \end{cases}$$

$(0,0)$

$$\begin{cases} x-7=0 \rightarrow x=7 \\ 0-14^2-4y^2=0 \end{cases}$$

$$\begin{cases} x=7 \\ y^2=-\dfrac{14^2}{4}<0 \end{cases}$$ IMP. in R $(y \in R)$

soluzione $(0,0)$

es.7: Calcolare in $\mathbb{C}$ tutte le soluzioni della seguente equazione:

(Tema d'esame 03-07-08)

$$z^4 = 3iz$$

$$z^4 - 3iz = 0$$

$$z(z^3 - 3i) = 0 \longrightarrow z_4 = 0$$

$$z^3 - 3i = 0 \qquad z^3 = 3i = 3e^{\frac{\pi}{2}i}$$

$\rho = 3$

$\vartheta = \frac{\pi}{2}$

$n = 3$

$$\vartheta_k = \frac{\vartheta + 2k\pi}{n}$$

$\vartheta_0 = \pi/6$ $\qquad z_0 \sqrt[3]{3}\, e^{\frac{\pi}{6}i}$

$\vartheta_1 = 5/6\,\pi$ $\qquad z_1 = \sqrt[3]{3}\, e^{\frac{5}{6}\pi i}$

$\vartheta_2 = \frac{3}{2}\pi$ $\qquad z_2 = \sqrt[3]{3}\, e^{3/2\,\pi i}$

es8: Determinare il luogo geometrico degli $z \in \mathbb{C}$ tali che:

(Teme d'esame 04-09.08)

$$7(z+\bar{z}) + z^2 = i + 14 \operatorname{Re} z \qquad z = x + iy$$

$$7(x+\cancel{iy} + x-\cancel{iy}) + (x+iy)^2 = i + 14x$$

$$\cancel{14x} + x^2 - y^2 + 2xyi - i - \cancel{14x} = 0$$

$$(x^2-y^2) + i(2xy-1) = 0$$

$$\begin{cases} x^2-y^2 = 0 \\ 2xy-1=0 \end{cases} \qquad \begin{cases} (x-y)(x+y)=0 \\ 2xy = 1 \end{cases}$$

$$\begin{cases} x-y = 0 \\ 2xy = 1 \end{cases} \quad \begin{cases} x=y \\ 2y^2=1 \end{cases} \qquad \begin{cases} x+y = 0 \\ 2xy = 1 \end{cases} \quad \begin{cases} x=-y \\ -2y^2=1 \end{cases}$$

IMP in R ($y \in R$)

$$\begin{cases} y = \pm \frac{1}{\sqrt{2}} = \pm \frac{\sqrt{2}}{2} \\ x = \pm \frac{\sqrt{2}}{2} \end{cases}$$

2 punti $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

$z_1 = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$; $z_2 = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i$

$(\vartheta_1 = \pi/4)$ $(\vartheta_2 = 5/4\,\pi)$

(Tema d'esame 04.05.08)

es. n°9: Scrivere in forma esponenziale le radici terze complesse di:

$$w = \frac{2\sqrt{2}}{(2-2i)(1-\sqrt{3}\,i)} = \frac{2\sqrt{2}}{2(1-i)(1-\sqrt{3}\,i)}$$

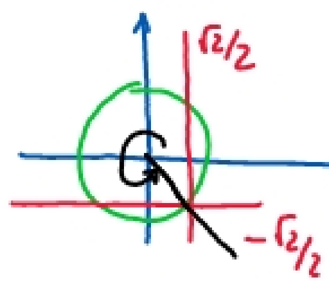
(si può "razionalizzare", o trasformare in notazione esponenziale)

$1 - i = \sqrt{2}\, e^{7/4 \pi i}$

$1 - \sqrt{3}\,i = 2\, e^{5/3 \pi i}$

$\rho_1 = \sqrt{2}$

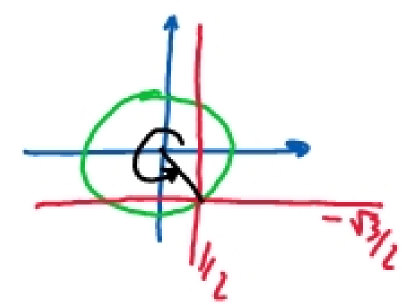
$$\vartheta_1 = \begin{cases} \cos\vartheta_1 = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \\ \sin\vartheta_1 = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2} \end{cases}$$



$\vartheta_1 = 7/4\, \pi$

$\rho_2 = \sqrt{4} = 2$

$$\vartheta_2 = \begin{cases} \cos\vartheta_2 = \frac{1}{2} \\ \sin\vartheta_2 = -\frac{\sqrt{3}}{2} \end{cases}$$



$\vartheta_2 = 5/3\, \pi$

$$w = \frac{\cancel{2}\cancel{\sqrt{2}}}{\cancel{2}\ \cancel{\sqrt{2}}\, e^{7/4 \pi i} \cdot 2\, e^{5/3 \pi i}} = \frac{1}{2}\, e^{-\frac{41}{12}\pi i} =$$

$$= \frac{1}{2}\, e^{7/12\, \pi}$$

l'angolo $-\frac{41}{12}\pi \rightarrow \frac{7}{12}\pi$

(3 semicirconferenze + una parte)

$$w = \frac{1}{2} e^{7/12 \pi i}$$

$$\rho = \frac{1}{2} \qquad \vartheta = \frac{7}{12}\pi \qquad \vartheta_k = \frac{\left(\frac{7}{12}\pi + 2k\pi\right)}{3}$$

$$\vartheta_0 = \frac{7}{12}\pi \cdot \frac{1}{3} = \frac{7}{36}\pi \qquad w_0 = \sqrt[3]{\frac{1}{2}}\; e^{\frac{7}{36}\pi i}$$

$$\vartheta_1 = \left(\frac{7}{12}\pi + 2\pi\right) \cdot \frac{1}{3} = \frac{31}{36}\pi \qquad w_1 = \sqrt[3]{\frac{1}{2}}\; e^{\frac{31}{36}\pi i}$$

$$\vartheta_2 = \left(\frac{7}{12}\pi + 4\pi\right) \cdot \frac{1}{3} = \frac{55}{36}\pi \qquad w_2 = \sqrt[3]{\frac{1}{2}}\; e^{\frac{19}{36}\pi i}$$

(N.B. $\frac{55}{36}\pi = \frac{36}{36}\pi + \frac{19}{36}\pi$)