SERIE E TAYLOR

es: $\sum_{n=1}^{+\infty} \left(e^{1/n^2} - 1\right)$

$a_n = e^{\frac{1}{n^2}} - 1 > 0 \quad \forall n \geq 1$

$b_n = \frac{1}{n^2}$

$\Rightarrow \lim_{n \to +\infty} \frac{e^{\frac{1}{n^2}} - 1}{1/n^2} = 1$

$\left(\lim_{x \to 0} \frac{e^x - 1}{x} = 1\right)$

CRITERIO DEL CONFRONTO $\quad a_n = e^{\frac{1}{n^2}} - 1 \simeq b_n = \frac{1}{n^2}$

$\Longrightarrow \sum a_n \rightarrow$ conv.

$b_n = \frac{1}{n^2} \quad \alpha > 1 \quad$ conv.

Sviluppo di Taylor $\quad e^x = 1 + x + \frac{x^2}{2} + \dots = \sum_{n=0}^{+\infty} \frac{x^n}{n!} \quad x \to 0^+$

$x = \frac{1}{n^2} \quad e^{1/n^2} = 1 + \frac{1}{n^2} + o\left(\frac{1}{n^2}\right) \Longrightarrow e^{\frac{1}{n^2}} - 1 = 1 + \frac{1}{n^2} - 1 \simeq \frac{1}{n^2}$

es $\displaystyle\sum_{n=2}^{+\infty} \frac{\log n}{n\sqrt{n+1}}$ serie data converge.

$$a_n = \frac{\log n}{n\sqrt{n+1}} = \frac{1}{n\sqrt{n+1}(\log n)^{-1}} \sim \frac{1}{n \cdot n^{1/2}(\log n)^{-1}} =$$

$\sqrt{n+1} \sim \sqrt{n} = n^{1/2}$

$$= \frac{1}{n^{3/2}(\log n)^{-1}}$$

$\alpha = 3/2$
$\beta = -1$

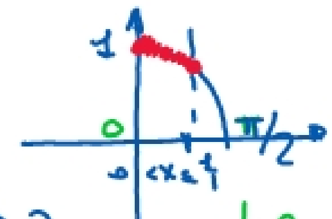
(*) sono soddisfatte

$$\sum \frac{1}{n^{\alpha}(\log n)^{\beta}} \quad \text{converge se} \quad \begin{cases} \alpha > 1 & \forall \beta \quad (*) \\ \alpha = 1 & \beta > 1 \end{cases}$$

es Sia $\alpha \in \mathbb{R}$, la serie $\displaystyle\sum_{n=1}^{+\infty} n^\alpha\left(1-\sqrt{\cos\frac{1}{n}}\right)$ converge se e solo se:

la serie è a termini positivi, infatti

$0<\frac{1}{n}\le 1 \Rightarrow 0<\cos\frac{1}{n}\le 1 \Rightarrow 1-\sqrt{\cos\frac{1}{n}}>0$



dal grafico del cos t

CRITERIO ASINTOTICO:

$$a_n = n^\alpha\left(1-\sqrt{\cos\frac{1}{n}}\right) \underset{\uparrow}{\sim} n^\alpha\left(1-\left(1-\frac{1}{2}\frac{1}{n^2}\right)^{1/2}\right) \simeq$$

sviluppando il $\cos x = \underbrace{1-\frac{1}{2!}x^2}+\frac{1}{4!}x^4 \dots$ per $x\to 0$

$$\left[1-\left(1-\frac{1}{2}\frac{1}{n^2}\right)\right]^{1/2} \sim (1+x)^\alpha \qquad \alpha=\frac{1}{2}$$

sviluppo
$$\begin{cases} (1-x)^\alpha = \underbrace{1-\alpha x}+\frac{\alpha(\alpha-1)}{2}x^2 \dots \quad \text{per } x\to 0 \\ (1+x)^\alpha = \underbrace{1+\alpha x}+\frac{\alpha(\alpha-1)}{2}x^2 \dots \end{cases}$$

$$\simeq n^\alpha\left(1-\left(1-\underset{\alpha}{\frac{1}{2}}\underbrace{\frac{1}{2n^2}}_{x}\right)\right) = n^\alpha\left(\cancel{1}-\cancel{1}+\frac{1}{2}\frac{1}{2n^2}\right) = \frac{n^\alpha}{4n^2} = \underbrace{\frac{1}{4}\left(\frac{1}{n}\right)^{2-\alpha}}$$

serie armonica generalizzata

convergenza se $2-\alpha>1 \iff \alpha<1$

$b_n = 1/n^{2-\alpha}$

es

Sia $\alpha \in \mathbb{R}$, studiare il carattere della serie:

$$\sum_{n=2}^{+\infty} \frac{e-\left(\frac{1}{n}+1\right)^n}{n^{\alpha+1}(\log n)^2}$$

• serie a termini positivi:

$\left(\frac{1}{n}+1\right)^n =$ succ. $\nearrow$; $\lim_{n\to+\infty}\left(\frac{1}{n}+1\right)^n = e \Rightarrow e > \left(1+\frac{1}{n}\right)^n$

• usando il criterio del confronto asintotico

$$\boxed{N} = e-\left(\frac{1}{n}+1\right)^n = e - e^{\log\left(\frac{1}{n}+1\right)^n} = e - e^{n\log\left(1+\frac{1}{n}\right)} = e - e^{1-\frac{1}{2n}} =$$

$$= e - e\cdot e^{-\frac{1}{2n}} =$$

$$= e\left[1-\underbrace{e^{-\frac{1}{2n}}}_{e^x}\right] = (*)$$

$b_n = n\underbrace{\log\left(1+\frac{1}{n}\right)}_{\text{sviluppo}} \simeq n\left[\frac{1}{n}-\frac{1}{2n^2}\right] = 1-\frac{1}{2n}$

$\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \dots$

$e^x = 1+x+\frac{x^2}{2!}$

$$(*) \simeq e\left[1-1+\frac{1}{2n}\right] = \boxed{e\cdot\frac{1}{2n}}$$

$$a_n = \frac{e - \left(\frac{1}{n}+1\right)^n}{n^{\alpha+1}(\log n)^2} \approx \frac{e \cdot \frac{1}{2n}}{n^{\alpha+1}(\log n)^2} = \frac{1}{n^{\alpha+2}(\log n)^2} \cdot \frac{e}{2} \sim$$

$$\sim \underbrace{\frac{1}{n^{\alpha+2}(\log n)^2}}_{\text{convergente}}$$

$$\sum_{n=2}^{+\infty} \frac{1}{n^{\alpha} \log^{\beta} n} \quad \text{se} \quad \begin{array}{ll} \forall \alpha > 1 & \forall \beta \in \mathbb{R} \\ \alpha = 1 & \forall \beta > 1 \end{array}$$

prendo le condizioni: $\beta = 2$

$$\alpha + 2 \geq 1 \implies \alpha \geq -1$$

quindi la serie converge $\iff \alpha \geq -1$

es determinare il carattere della serie

$$\sum_{n=1}^{+\infty} \frac{\sin(n^5) - \sqrt{n}}{\sqrt{n+\log n}\,\left(\log(n^4+n!)\right)}$$

si osserva che

$$\underbrace{\sin n^5}_{\sin x \le 1 \ \forall x \in \mathbb{R}} - \sqrt{n} < 0 \qquad \forall n \geq 1$$

allora la serie data è a termini negativi, per poter applicare il criterio del confronto (o asintotico) occorre che la serie sia a termini non negativi, per cui può essere scritta:

$$\sum_{n=1}^{+\infty} a_n = \sum_{n=1}^{+\infty} \frac{\sin(n^5) - \sqrt{n}}{\sqrt{n+\log n}\,\left(\log n^4+n!\right)} = -\sum_{n=1}^{+\infty} \frac{\sqrt{n} - \sin(n^5)}{\sqrt{n+\log n}\,\left(\log n^4+n!\right)}$$

con $a_n = -b_n$ e $b_n \geq 0 \quad \forall n \in \mathbb{N}$

quindi si studia il carattere della serie $\sum_{n=1}^{+\infty} b_n$

$$b_n = \frac{\sqrt{n} - \sin n^5}{\sqrt{n+\log n}\,\left(\log(n^n+n!)\right)} =$$

- $\sqrt{n} - \sin n^5 \sim \sqrt{n}$ $\qquad \lim\limits_{n\to+\infty} \dfrac{\sqrt{n} - \overset{[-1,1]}{\sin n^5}}{\sqrt{n}} = 1$ $\qquad \left(\text{Racc: } \dfrac{\sqrt{n}\left(1 - \frac{\sin n^5}{\sqrt{n}}\right)}{\sqrt{n}}\right)$, $\dfrac{\sin n^5}{\sqrt{n}} \to 0$

- $\sqrt{n+\log n} \sim \sqrt{n}$ $\qquad \lim\limits_{n\to+\infty} \dfrac{\sqrt{n+\log n}}{\sqrt{n}} = 1$ $\qquad \left(\dfrac{\sqrt{n\left(1+\frac{\log n}{n}\right)}}{\sqrt{n}}\right)$, $\dfrac{\log n}{n} \to 0$

- $\log(n^n+n!) \sim \log n^n = n\log n$ $\qquad \lim\limits_{n\to+\infty} \dfrac{\log(n^n+n!)}{\log n^n} = 1$ $\qquad \left(\dfrac{\log\left(n^n\left(1+\frac{n!}{n^n}\right)\right)}{\log n^n} = \dfrac{\log n^n + \log\left(1+\frac{n!}{n^n}\right)}{\log n^n}\right)$, $\dfrac{n!}{n^n} \to 0$

$$b_n \simeq \frac{\cancel{\sqrt{n}}}{\cancel{\sqrt{n}}\,\log n^n} = \frac{1}{n\log n}$$

$$\sum_{n=1}^{+\infty} b_n \sim \sum_{n=2}^{+\infty} \frac{1}{n\log n} = +\infty$$

$$\sum_{n=1}^{+\infty} a_n = -\sum_{n=1}^{+\infty} b_n = -\infty \quad \text{diverge negativamente}$$

es determinare il carattere della serie

$$\sum_{n=3}^{+\infty} \frac{n!}{(2+(n+1)!)^\alpha} \qquad \text{con } \alpha \in R$$

$$a_n = \frac{n!}{(2+(n+1)!)^\alpha} = \frac{n!}{[(n+1)!]^\alpha \underbrace{\left[1+\underset{\to 0}{\frac{2}{(n+1)!}}\right]^\alpha}_{1^\alpha \sim 1}} \simeq$$

$$\simeq \frac{n!}{[(n+1)\ n!]^\alpha} = \frac{n!}{(n+1)^\alpha \cdot (n!)^\alpha} =$$

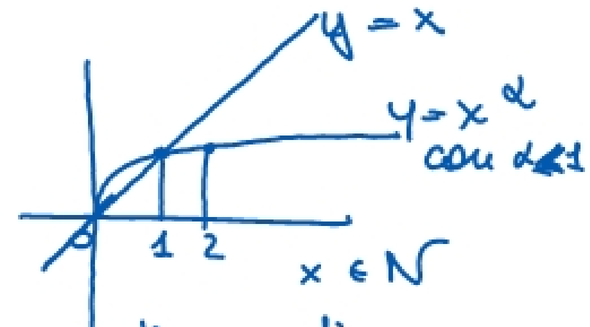
$$= \frac{1}{(n+1)^\alpha \cdot (n!)^{\alpha-1}}$$

convergenza quando:

$\alpha = 1 \qquad a_n = \frac{1}{(n+1)} \qquad$ divergente $\left(\sim \frac{1}{n}\right)$

$\alpha > 1 \qquad a_n \sim \frac{1}{(n+1)^\alpha (n!)^{\alpha-1}} < \left(\frac{1}{n+1}\right)^\alpha \qquad$ convergente se si parte dall'analisi dei termini:

per $\alpha \le 1$ si ha $\left(2+(n+1)!\right)^{\alpha} \le 2+(n+1)!$



infatti $x^{\alpha} \le x \quad \forall x \geq 2 \quad \forall \alpha \le 1$

da cui

$a_n \geq \dfrac{n!}{2+(n+1)!} \sim \dfrac{1}{n}$ (serie armonica divergente

quindi $\sum a_n$ diverge per $\alpha \le 1$

per $\alpha > 1 \Rightarrow \left(2+(n+1)!\right)^{\alpha} > \left((n+1)!\right)^{\alpha} = (n+1)^{\alpha} \cdot (n!)^{\alpha} \geq n!(n+1)^{\alpha}$

quindi $a_n \le \dfrac{n!}{n!\,(n+1)^{\alpha}} \sim \left(\dfrac{1}{n}\right)^{\alpha}$ serie armonica generalizzata converge $\Leftrightarrow \alpha > 1$

allora $\sum a_n$ converge per $\alpha > 1$

es Trovare l'estremo inferiore dell'insieme:

$$A = \left\{ \alpha \in \mathbb{R} : \sum_{n=3}^{+\infty} \left(\frac{n^2 + \sin n}{n^\alpha \log n} \right)^{1/2} < +\infty \right.$$

La serie data è a termini positivi

infatti $\left.\begin{array}{ll} n^2 + \sin n \geq n^2 - 1 > 0 & \forall n \geq 3 \\ \log n > 0 & \forall n \geq 3 \end{array}\right\} \Rightarrow a_n = \left(\frac{n^2 + \sin n}{n^\alpha \log n} \right)^{1/2} > 0 \quad \forall n \geq 3$

Usando il criterio del confronto asintotico.

$$\lim_{n \to +\infty} \frac{\left(\frac{n^2 + \sin n}{n^\alpha \log n} \right)^{1/2}}{\left(\frac{n^2}{n^\alpha \log n} \right)^{1/2}} = 1 \Rightarrow a_n \sim \left(\frac{n^2}{n^\alpha \log n} \right)^{1/2} = \frac{1}{n^{\frac{\alpha - 2}{2}} \log n^{1/2}}$$

considerando $\sum_{n=2}^{+\infty} \frac{1}{n^\beta \log^\gamma n}$ conv. $\begin{cases} \beta > 1, \forall \gamma \\ \beta = 1, \forall \gamma > 1 \end{cases}$

converge $\Leftrightarrow$

$$\underbrace{\frac{\alpha - 2}{2}}_{\beta} > 1 \Leftrightarrow \alpha - 2 > 2 \Rightarrow \alpha > 4$$

$$\gamma = 1/2 < 1$$

$\Rightarrow A = (4, +\infty) \qquad \inf A = 4$

es (T.E) Sia $\alpha \in \mathbb{R}$, la serie numerica

$$\sum_{n=2}^{+\infty} \left(\frac{e^{\alpha n}}{n^2} + \frac{\log(n+1) - \log n}{n^{\alpha+2} \log n} \right)$$

converge se :

Sia: $\displaystyle\sum_{n=2}^{+\infty} a_n = \underbrace{\sum_{n=2}^{+\infty} \frac{e^{\alpha n}}{n^2}}_{\Sigma(1)} + \underbrace{\sum_{n=2}^{+\infty} \frac{\log(n+1) - \log n}{n^{\alpha+2} \log n}}_{\Sigma(2)}$ converge $\Leftrightarrow$ (1) e (2) conv.

(1) $\displaystyle\sum_{n=2}^{+\infty} \frac{e^{\alpha n}}{n^2}$: C. N. $\displaystyle\lim_{n\to+\infty} \frac{e^{\alpha n}}{n^2} = 0 \Leftrightarrow \alpha \leq 0$

quindi per $\alpha \leq 0$ $\quad \dfrac{e^{\alpha n}}{n^2} < \dfrac{1}{n^2}$ serie armonica generalizzata che converge $(\alpha = 2)$

per il criterio del confronto $\Sigma(1)$ converge per $\alpha \leq 0$

(2) $\displaystyle\sum_{n=2}^{+\infty} \frac{\log(n+1)-\log n}{n^{\alpha+2}\log n} = \sum_{n=2}^{+\infty} \frac{\log\frac{n+1}{n}}{n^{\alpha+2}\log n} = \sum_{n=2}^{+\infty} \frac{\overbrace{\log\left(1+\frac{1}{n}\right)}^{\sim\frac{1}{n}}}{n^{\alpha+2}\log n} = \sum b_n$

C.N. $\displaystyle\lim_{n\to+\infty} b_n = 0 \iff \frac{\log(1+\frac{1}{n})}{n^{\alpha+2}\log n} \simeq \frac{1/n}{n^{\alpha+2}\log n} \simeq \frac{1}{n^{\alpha+3}\log n}$

$\log(1+x) \sim x$ per $x \to 0$

$\iff \alpha+3 \geq 0 \implies \alpha \geq -3$

per $\alpha \geq -3$ otteniamo la convergenza

$\displaystyle b_n \simeq \frac{1}{n^{\alpha+3}\log n}$ confrontando con $\displaystyle\sum \frac{1}{n^{\beta}(\log n)^{\alpha}}$

conv: $\begin{cases} \beta > 1 & \forall \alpha \\ \beta = 1 & \forall \alpha > 1 \end{cases}$

$\implies \alpha+3 > 1 \implies \alpha > -2$

$\sum b_n$ converge se $\alpha > -2$

$\implies$ considerando (1) e (2) $\begin{cases} \alpha > -2 \\ \alpha \leq 0 \end{cases} \implies$ convergenza della serie nel suo complesso se $-2 < \alpha \leq 0$